

A Euclidean formulation of relativistic quantum mechanics

P. Kopp and W. N. Polyzou

Department of Physics and Astronomy, The University of Iowa, Iowa City, IA 52242

(Dated: June 22, 2011)

In this paper we discuss a formulation of relativistic quantum mechanics that uses Euclidean Green functions or generating functionals as input. This formalism has a close relation to quantum field theory, but as a theory of linear operators on a Hilbert space, it has many of the advantages of quantum mechanics. One interesting feature of this approach is that matrix elements of operators in normalizable states on the physical Hilbert space can be calculated directly using the Euclidean Green functions without performing an analytic continuation. The formalism is summarized in this paper. We discuss the motivation, advantages and difficulties in using this formalism. We discuss how to compute bound states, scattering cross sections, and finite Poincaré transformations without using analytic continuation. A toy model is used to demonstrate how matrix elements of $e^{-\beta H}$ in normalizable states can be used to construct-sharp momentum transition matrix elements.

PACS numbers: 21.45+v

I. MOTIVATION

In this paper we investigate a framework for constructing relativistic quantum mechanical models of few-degree-of-freedom systems that are inspired by an underlying quantum field theory. Our interest is physics at the few GeV energy scale. Poincaré invariance is an important symmetry at this scale since the energies are comparable to the mass scale of hadrons. Most models of systems at these energies are motivated by quantum field theory, but their connection with quantum mechanical models of a finite number of degrees of freedom is not straightforward. The advantage of a quantum mechanical model of a finite number of degrees of freedom is that the theory is linear and it can be in-principle solved, even for strongly interacting systems, with mathematically controlled errors. Bound systems of particles present no special problems; they are just point-spectrum eigenstates of the mass operator (rest energy). The construction of a unitary multichannel S operator can be performed using the same methods that are used in non-relativistic models.

One approach that has been successfully used to formulate realistic Poincaré invariant quantum mechanical models of few-hadron systems is Poincaré invariant quantum mechanics[1]. In this approach a dynamical unitary representation of the Poincaré group is constructed on a few-particle Hilbert space. This approach has been successfully applied to treat a number of few-hadron or few quark problems. The virtue of this approach is that it is possible to construct quantum mechanical models with a finite number of degrees of freedom that have a unitary representation of the Poincaré group, satisfy a spectral condition, and for fixed numbers of particles satisfy cluster separability. These are essentially all of the axioms of quantum field theory, except microscopic locality, which cannot be tested experimentally and requires an infinite number of degrees of freedom. One of the disadvantages of this approach is that the models do not have a straightforward relation to a Lagrangian field theory. This makes it difficult to use a field theory like QCD to systematically improve or constrain the models.

Because of the difficulties discussed in the previous paragraph it is desirable to explore alternate formulations of relativistic quantum mechanics that have a more direct relation to Lagrangian field theory while preserving the structure of the underlying quantum theory. The alternative that we pursue in this work is to formulate models that are motivated by the standard reconstruction of a quantum theory from the field theory. The Euclidean formulation of quantum field theory is convenient for this purpose because (1) it has a direct relation to the action or Lagrangian through either formal path integrals or the Dyson expansion and (2) it is possible to use the Euclidean Green functions or generating functional to directly construct the physical Hilbert space using the Euclidean reconstruction theorem. In this paper we argue that using this framework it is possible to compute all interesting quantum mechanical observables without explicit analytic continuation.

The proposed models are constructed by restricting the number of degrees of freedom. In passing to a few-body model all of the axioms of field theory cannot be preserved. One of the attractive features of the Euclidean reconstruction theorem is that locality is an independent requirement that can be relaxed without violating the other axioms[2]. A feature of this approach is that the dynamics is introduced directly by the model Euclidean Green functions or generating functional, rather than in a model Hamiltonian or Lagrangian. One of the challenges of this approach is to find a robust framework for modeling Euclidean Green functions or generating functionals with the required properties. An advantage is that standard Euclidean field theoretic methods can be used to formulate models.

The construction of the physical Hilbert space of the field theory from a Euclidean generating functional is discussed in the next section. The generating functional for a scalar field is used to illustrate the basic construction. The

construction of the Poincaré Lie algebra is discussed in the third section. The resulting generators are self-adjoint operators on the physical Hilbert space. The construction of one-particle states and the computation of finite Poincaré transformations on these states is discussed in section four. The one-particle states are used to construct scattering states in section five. The computation of finite Poincaré transformations on the scattering states are also discussed in this section. Everything discussed in sections 2-5 uses only Euclidean generating functionals and Euclidean test functions. Section six discuss how the results of sections 2-5 are expressed in terms of Euclidean Green functions. This is more practical for formulating models. Models of finite number of degrees of freedom are discussed in section seven by considering the structure of models of relativistic nucleon-nucleon scattering. Numerical tests of the proposed method to compute scattering observables from matrix elements of $e^{-\beta H}$ in normalizable states are given in section eight using an exactly solvable non-relativistic model. The results suggest that these methods can be used to compute cross sections at the few GeV energy scale. Our conclusions and the outlook for this approach are discussed in the last section.

II. QUANTUM MECHANICS - HILBERT SPACE REPRESENTATION

To illustrate the construction of the model quantum theory we assume that we are given an Euclidean generating functional $Z[f]$ associated with a scalar field. For the purpose of illustration we assume that $Z[f]$ has all of the properties that are expected of the generating functional of a scalar field theory. These properties include Euclidean invariance, reflection positivity, reality, and cluster separability.

The generating functional relates the Lagrangian of the field theory to the Hilbert space formulation of the theory. In Lagrangian field theory the generating functional is formally the functional Fourier transform of an Euclidean path-integral measure with action $A[\phi]$

$$Z[f] = \frac{\int D[\phi] e^{-A[\phi] + i\phi(f)}}{\int D[\phi] e^{-A[\phi]}} = \sum_n \frac{(i)^n}{n!} S_n(\underbrace{f, \dots, f}_{n \text{ times}}) = \exp\left(\sum_n \frac{(i)^n}{n!} S_n^c(\underbrace{f, \dots, f}_{n \text{ times}})\right) \quad (2.1)$$

where $f = f(\tau, \mathbf{x})$ is a suitable Euclidean test function and S_n is the n -point Euclidean Green function, and S_n^c is the connected n -point Euclidean Green function. In this approach the generating functional $Z[f]$ is the dynamical input that replaces the Lagrangian or Hamiltonian. A connection with an underlying Lagrangian or path integral is not required, however this connection is an important source of phenomenology, which is the primary motivation for developing this formalism.

In what follows Euclidean space-time coordinates are denoted by $\mathbf{x} = (\tau, \mathbf{x})$. Euclidean invariance of the generating functional means that

$$Z[f_{E,a}] = Z[f]. \quad (2.2)$$

for

$$f_{E,a}(\mathbf{x}) = f(E^{-1}(\mathbf{x} - \mathbf{a})), \quad (2.3)$$

where E is an $O(4)$ rotation and \mathbf{a} is a Euclidean space-time displacement.

The construction of the physical Hilbert space from Euclidean Green functions was given by Osterwalder and Schrader [2][3]. A simpler construction in terms of the generating functional was given by Fröhlich[4]. We use Fröhlich's approach to illustrate the construction using the generating functional for a "scalar field". To construct the physical Hilbert space Osterwalder and Schrader select a Euclidean time axis and restrict the space of test functions, $f(\tau, \mathbf{x})$, to real-valued Schwartz functions of four Euclidean variables that vanish for negative Euclidean times;

$$\mathcal{S}_+ = \{f(\tau, \mathbf{x}) \in \mathcal{S} | f(\tau, \mathbf{x}) = 0 \quad \tau < 0\}. \quad (2.4)$$

We call elements of \mathcal{S}_+ positive-time test functions. In what follows all test functions will be assumed to be positive-time test functions unless stated otherwise.

Osterwalder and Schrader introduce the Euclidean time-reflection operator Θ defined by

$$\Theta f(\tau, \mathbf{x}) = f(-\tau, \mathbf{x}). \quad (2.5)$$

The generating functional, $Z[f]$, is reflection positive if for any finite sequence of real positive-time test functions, $\{f_1 \cdots f_n\} \in \mathcal{S}_+$, the $n \times n$ matrix

$$M_{ij} := Z[f_i - \Theta f_j] \geq 0; \quad (2.6)$$

is non-negative. This condition is not automatic. It holds for generating functionals for free fields and for some lattice truncations of interacting theories. In general it is a requirement on physically acceptable models. In what follows we simply assume this condition is satisfied.

In Fröhlich's construction a dense set of normalizable vectors in the physical Hilbert space is represented by complex wave-functionals of the form

$$B[\phi] = \sum_{j=1}^{n_b} b_j e^{i\phi(f_j)} \quad (2.7)$$

where b_i are complex numbers and $f_j(\tau, \mathbf{x})$ are real positive-time test functions. The argument ϕ can be thought of as an abstract integration variable. This interpretation is motivated by the path integral representation of the generating functional.

The physical scalar product of two such wave-functionals, $B[\phi]$, and

$$C[\phi] = \sum_{k=1}^{n_c} c_k e^{i\phi(g_k)} \quad (2.8)$$

is given directly in terms of the generating functional by

$$\langle B|C \rangle := \sum_{j,k}^{n_b, n_c} b_j^* c_k Z[g_k - \Theta f_j]. \quad (2.9)$$

Reflection positivity, (2.6), is equivalent to the statement

$$\langle B|B \rangle \geq 0. \quad (2.10)$$

The physical Hilbert space is obtained by identifying wave functions whose difference has zero norm and completed by adding convergent sequences of wave functionals.

The inner product (2.9) is the physical quantum mechanical scalar product, even though the input only involves the Euclidean generating functional and positive-time Euclidean test functions! No analytic continuation is used. An explicit illustration of this relationship is given in section six for free scalar particles in eq. (6.4) and free spin 1/2 particles in eq. (6.5).

While the computation of the exact generating functional is equivalent to solving the field theory, models of generating functionals are easily constructed. For example, consider the representation of the generating functional in terms of connected Euclidean Green functions. If we isolate the contribution to the generating functional from the connected n -point Green function (see 2.1),

$$Z_n^c[f] := e^{\frac{(i^n)}{n!} S_n^c(f, \dots, f)}, \quad (2.11)$$

then the full generating functional is the product

$$Z[f] = \prod_n Z_n^c[f]. \quad (2.12)$$

It follows that the matrix that gives the quantum mechanical scalar product, (2.6), has the decomposition

$$M_{ij} := Z[f_i - \theta f_j] = \prod_n Z_n^c[f_i - \theta f_j] = \prod_n M_{ij}^n \quad (2.13)$$

which is an infinite Schur product of the matrices M_{ij}^n . A sufficient condition for positivity of M_{ij} is that each M_{ij}^n in the Schur product is positive (this is because the Schur product can be expressed as the restriction of the tensor product of positive operators to the diagonal subspace).

Thus, one strategy for constructing models is to use the representation (2.13), to build models starting with the free field two-point function. $Z_2^c[f]$ is reflection positive if it is the generating functional for a generalized free field. Including a reflection positive $Z_4^c[f]$ in the Schur product (2.13) would give a generating functional for an interacting many-body theory, where the dynamical input is a suitable connected Euclidean 4-point function,

$$Z_{model}[f] = Z_2^c[f] Z_4^c[f]. \quad (2.14)$$

Positivity of M_{ij}^4 means that that

$$Z_4^c[f_i - \theta f_j] = e^{\frac{i^4}{4!} S_4^c(f_i - \theta f_j, f_i - \theta f_j, f_i - \theta f_j, f_i - \theta f_j)} \quad (2.15)$$

is a positive matrix for any finite sequence of positive-time test functions. The model can be refined by including additional factors, $Z_n^c[f]$.

III. RELATIVISTIC INVARIANCE

Poincaré invariance of a quantum theory implies the existence of a unitary representation of the Poincaré group on the physical Hilbert space [5]. Equivalently there should be a set of ten infinitesimal generators of the Poincaré group represented by self-adjoint operators satisfying the Poincaré commutation relations.

The relation between the complex Euclidean group and the complex Poincaré group is relevant for constructing Poincaré generators. To understand the connection between the 4-dimensional Euclidean group and the Poincaré group consider the matrices

$$X = \begin{pmatrix} t - z & x - iy \\ x + iy & t + z \end{pmatrix} \quad \mathbf{X} = \begin{pmatrix} i\tau - z & x - iy \\ x + iy & i\tau + z \end{pmatrix}. \quad (3.1)$$

A simple calculation shows that $\det(X) = t^2 - \mathbf{x}^2$ and $\det(\mathbf{X}) = -(\tau^2 + \mathbf{x}^2)$, which are the Minkowski and Euclidean invariant (distances)² respectively. Both determinants are preserved under the linear transformations

$$X \rightarrow X' = AXB^t \quad \mathbf{X} \rightarrow \mathbf{X}' = A\mathbf{X}B^t \quad \det(A) = \det(B) = 1 \quad (3.2)$$

where A and B are complex matrices with unit determinant. In general the pair (A, B) defines a complex Lorentz or complex $O(4)$ transformation. Both (A, B) and $(-A, -B)$ correspond to the same linear transformation of the coordinates. In general, the transformed coordinates may become complex, but the determinant remains real and unchanged. If $B = A^*$ then transformation $X' = AXB^t$ is a real Lorentz transformation; if $A, B \in SU(2)$ then the transformation $\mathbf{X}' = A\mathbf{X}B^t$ is a real $O(4)$ transformation.

When $A, B \in SU(2)$ the transformation $X' = AXB^t$ is a six parameter subgroup of the complex Lorentz group. While this represents a real Euclidean transformation on the Euclidean Hilbert space (without the Euclidean time reversal Θ), it defines a complex Lorentz transformation on the physical Hilbert space. It is possible to extract the ten Poincaré generators by considering infinitesimal forms of these complex Lorentz transformations.

In order to get a unitary representation of the real Poincaré group on the physical Hilbert space the generators must be self-adjoint. It turns out that the Θ in the physical scalar product breaks Euclidean invariance in just the right way to ensure that all of the Poincaré generators are self-adjoint.

We begin by defining real Euclidean transformations on wave functionals by

$$(T(E, \mathbf{a})B)[\phi] = \sum_{j=1}^{n_b} b_j e^{i\phi(f_{j,E,\mathbf{a}})} \quad f_{j,E,\mathbf{a}}(\mathbf{x}) := f_j(E^{-1}(\mathbf{x} - \mathbf{a})). \quad (3.3)$$

These transformations leave the generating functional invariant, $Z[f] = Z[f_{E,\mathbf{a}}]$. In general they will not preserve the positive time constraint.

Before we use these transformations to construct Poincaré generators on the physical Hilbert space, it is useful to note that the wave functionals can also be considered as multiplication operators. For example the operator $B[\phi]$ acting on the wave functional $C[\phi]$ is a wave functional $D[\phi]$ defined by

$$D[\phi] := B[\phi]C[\phi] = \sum_{j=1}^{n_b} \sum_{k=1}^{n_c} b_j c_k e^{i\phi(f_j + g_k)}. \quad (3.4)$$

This will become important when we formulate scattering asymptotic conditions. Note that in the wave functional representation the order of the Minkowski field operators is determined by the order of the Euclidean time support of the positive time Schwartz functions.

Next we consider the real Euclidean transformations, $T(E, \mathbf{a})$, as complex Poincaré transformations on the physical Hilbert space. It is useful to treat pure rotations, space translations, Euclidean time translations, and Euclidean rotations in planes containing the time axis separately. The Euclidean time reversal Θ operator does not commute with the last two transformations.

We define action of the Poincaré generators on the wave functionals considered as operators by:

$$[\mathbf{P}, B[\phi]] = -i \frac{\partial}{\partial \mathbf{a}} [T(I, (0, \mathbf{a})), B[\phi]]|_{\mathbf{a}=0} := -i \sum_{j=1}^{n_b} b_j \frac{\partial}{\partial \mathbf{a}} e^{i\phi(f_{j,I,(0,\mathbf{a})})}|_{\mathbf{a}=0}, \quad (3.5)$$

$$[\mathbf{J} \cdot \hat{\mathbf{n}}, B[\phi]] = -i \frac{\partial}{\partial \xi} (T(R(\hat{\mathbf{n}}, \xi), 0)B[\phi])|_{\xi=0} := -i \sum_{j=1}^{n_b} b_j \frac{\partial}{\partial \xi} e^{i\phi(f_{j,(R(\hat{\mathbf{n}}, \xi), 0)})}|_{\xi=0}. \quad (3.6)$$

where $R(\hat{\mathbf{n}}, \xi)$ is an ordinary rotation about the $\hat{\mathbf{n}}$ axis by an angle ξ . For the Hamiltonian we require that $\beta > 0$ in $T(I, (\beta, a))$ to preserve the positive-time support condition:

$$[H, B[\phi]] = -\frac{\partial}{\partial \beta} (T(I, (\beta, 0))B)[\phi]_{|\beta=0} := -\sum_{j=1}^{n_b} b_j \frac{\partial}{\partial \beta} e^{i\phi(f_j, I, (\beta, 0))}. \quad (3.7)$$

For the boost generators we first restrict the support of the test functions f_j in the wave functionals to a cone symmetric about the positive Euclidean time axis that makes an angle $0 \leq \chi < \pi/2$ with the Euclidean time axis

$$\mathcal{S}_{\chi,+} := \{f \in \mathcal{S}_+ \mid f(\tau, \mathbf{x}) = 0 \quad \tan^{-1}\left(\frac{\tau}{|\mathbf{x}|}\right) \geq \chi\}. \quad (3.8)$$

This ensures that the support condition is preserved for sufficiently small rotations. On these wave functionals we consider the Euclidean rotation $T(R_e(\hat{\mathbf{n}}, \xi), 0)$ in the $\hat{\mathbf{n}} - \tau$ plane through angle $\xi < \pi/2 - \chi$:

$$f_{j,\xi,\hat{\mathbf{n}}}(\tau, \mathbf{x}) := f_j(\tau', \mathbf{x}') \quad f_j \in \mathcal{S}_{\chi,+}, \quad (3.9)$$

with

$$\tau' = \tau \cos(\xi) - x_{\hat{\mathbf{n}}} \sin(\xi) \quad x'_{\hat{\mathbf{n}}} = x_{\hat{\mathbf{n}}} \cos(\xi) + \tau \sin(\xi). \quad (3.10)$$

The restrictions on the parameters ξ and χ ensure that initial and final vectors are in the physical Hilbert space. On these vectors the rotationless boost generator is defined by

$$[\mathbf{K} \cdot \hat{\mathbf{n}}, B[\phi]] = -\frac{\partial}{\partial \xi} (T(R_e(\hat{\mathbf{n}}, \xi), 0)B)[\phi]_{|\xi=0} := -\sum_{j=1}^{n_b} b_j \frac{\partial}{\partial \xi} e^{i\phi(f_j, (R_e(\hat{\mathbf{n}}, \xi), 0))} \quad (3.11)$$

where $R_e(\hat{\mathbf{n}}, \xi)$ is the Euclidean space-time rotation (3.10). Note the absence of the i in the expressions for H and \mathbf{K} . This is compensated for by the Θ that appears in the physical scalar product.

Direct calculations show that the ten operators $H, \mathbf{P}, \mathbf{J}, \mathbf{K}$ satisfy the Poincaré commutation relations and are formally Hermitian on the physical Hilbert space. Self-adjointness of $H, \mathbf{P}, \mathbf{J}$ follows because these operators are generators of either one parameter unitary groups or a contractive Hermitian semigroup. The contractive nature of time Euclidean time evolution is proved using reflection positivity and positivity properties of the generating functional[6]. This also ensures that Hamiltonian satisfies the spectral condition:

$$H \geq 0. \quad (3.12)$$

Self-adjointness of \mathbf{K} can be established by verifying that it is the generator of a local symmetric semigroup [7][8][9].

Matrix elements of the Poincaré generators in normalizable states can be expressed directly in terms of the generating functional

$$\langle B|H|C \rangle = -\frac{\partial}{\partial \beta} \left(\sum_{j=1}^{N_b} \sum_{k=1}^{N_c} b_j^* c_k Z[g_{k,I,(\beta,0)} - \Theta f_j] \right)_{\beta=0}, \quad (3.13)$$

$$\langle B|\mathbf{P}|C \rangle = -i \frac{\partial}{\partial \mathbf{a}} \left(\sum_{j=1}^{n_b} \sum_{k=1}^{n_c} b_j^* c_k Z[g_{k,I,(0,\mathbf{a})} - \Theta f_j] \right)_{\mathbf{a}=0}, \quad (3.14)$$

$$\langle B|\hat{\mathbf{n}} \cdot \mathbf{J}|C \rangle = -i \frac{\partial}{\partial \xi} \left(\sum_{j=1}^{b_b} \sum_{k=1}^{n_c} c_j^* d_k Z[g_{k,R(\hat{\mathbf{n}},\xi),0} - \Theta f_j] \right)_{\xi=0}, \quad (3.15)$$

$$\langle B|\hat{\mathbf{n}} \cdot \mathbf{K}|C \rangle = -\frac{\partial}{\partial \xi} \left(\sum_{j=1}^{b_b} \sum_{k=1}^{n_c} c_j^* d_k Z[g_{k,R_e(\hat{\mathbf{n}},\xi),0} - \Theta f_j] \right)_{\xi=0}. \quad (3.16)$$

The formal Hermiticity of the generators defined above can be deduced from these expressions. For example

$$\begin{aligned}
\langle B|H^\dagger|C\rangle &= \langle C|H|B\rangle^* = \\
&= -\frac{\partial}{\partial\beta} \left(\sum_{j=1}^{N_b} \sum_{k=1}^{N_c} b_j c_k^* Z[-f_{k,I,(\beta,0)} + \Theta g_j] \right)_{\beta=0} = \\
&= -\frac{\partial}{\partial\beta} \left(\sum_{j=1}^{N_b} \sum_{k=1}^{N_c} b_j c_k^* Z[-f_k + (\Theta g)_{j,I,(-\beta,0)}] \right)_{\beta=0} = \\
&= \left(\sum_{j=1}^{N_b} \sum_{k=1}^{N_c} b_j c_k^* Z[\Theta g_{j,I,(\beta,0)} - f_k] \right)_{\beta=0} = \langle B|H|C\rangle
\end{aligned} \tag{3.17}$$

where we have used reality, $Z^*[f] = Z[-f]$, Euclidean invariance, and properties of Θ . Hermiticity of the rotationless boost generators follows using the same argument. The Euclidean time-reversal operator, Θ , plays the role of the missing factor of i when integrating by parts.

The commutation relations can be verified by explicit computation, however they also follow as a direct consequence of the relation between complex $O(4)$ and the complex Lorentz group.

Matrix elements of $e^{-\beta H}$ can also be directly computed in terms of the generating functional:

$$\langle B|e^{-\beta H}|C\rangle = \sum_{j=1}^{N_b} \sum_{k=1}^{N_c} b_j^* c_k Z[g_{k,I(\beta,0)} - \Theta f_j]. \tag{3.18}$$

These matrix elements are needed to compute scattering cross sections and only involve elementary quadratures.

Matrix elements of the mass Casimir operator can be expressed in terms of the Poincaré generators

$$M^2 = H^2 - \mathbf{P}^2 \tag{3.19}$$

$$\langle B|M^2|C\rangle := \left(\frac{\partial^2}{\partial\beta^2} + \frac{\partial^2}{\partial\mathbf{a}^2} \right) \sum_{j=1}^{N_b} \sum_{k=1}^{N_c} b_j^* c_k Z[g_{k,I(\beta,\mathbf{a})} - \Theta f_j]_{|\beta=\mathbf{a}=0}. \tag{3.20}$$

Finally we note that the real Euclidean transformations, $T(E, \mathbf{a})$ can be expressed in terms of the Poincaré generators on the physical Hilbert space by

$$T(I, (\beta, \mathbf{a})) = e^{-\beta H + i\mathbf{a} \cdot \mathbf{P}} \quad T(R(\hat{\mathbf{n}}, \psi), 0) = e^{i\mathbf{J} \cdot \hat{\mathbf{n}} \psi} \quad T(R_e(\hat{\mathbf{n}}, \psi), 0) = e^{\mathbf{K} \cdot \hat{\mathbf{n}} \psi}. \tag{3.21}$$

Thus Euclidean time evolution and rotations in Euclidean space-time planes look like imaginary time evolution and Lorentz transformation with imaginary rapidities.

The operators defined in (3.5, 3.6, 3.7 and 3.11) are self-adjoint operators on the physical Hilbert space that satisfy the Poincaré commutation relations. Formally they can be exponentiated to give a unitary representation of the Poincaré group on the physical Hilbert space but, as we will see in the next two sections, this exponentiation is never needed.

The expressions for the matrix elements of all of the Poincaré generators in normalizable states (3.13-3.16) are directly expressed in terms of the Euclidean generating functional and Euclidean test functions. Analytic continuation is not used.

IV. PARTICLES

Given the dense set of wave functionals of the form (2.7) and the physical scalar product (2.9) the Gram-Schmidt method can be formally used to construct a complete orthonormal set of wave functionals $B_n[\phi]$:

$$\langle B_n|B_m\rangle = \delta_{mn}. \tag{4.1}$$

Since the orthonormal wave functionals are complete, normalizable one-particle states are linear combinations of these orthonormal wave functionals with square summable coefficients:

$$\Psi_\lambda[\phi] = \sum_n b_n B_n[\phi] \quad \sum_n |b_n|^2 < \infty \quad (4.2)$$

that are eigenstates of the mass square Casimir operator (3.19) of the Poincaré group with eigenvalue λ^2 in the point spectrum:

$$\sum_n \langle B_m | M^2 | B_n \rangle b_n = \sum_n \langle B_m | (H^2 - \mathbf{P}^2) | B_n \rangle b_n = \lambda^2 \delta_{mn} b_n. \quad (4.3)$$

These normalizable states are infinitely degenerate because there is an associated wave packet in the particle's momentum and spin. For suitable wave packets these normalizable eigenstates can be decomposed into simultaneous eigenstates of mass and linear momentum using translations and Fourier transforms:

$$\Psi_{\lambda, \mathbf{p}}[\phi] = \int \frac{d\mathbf{a}}{(2\pi)^{3/2}} e^{-i\mathbf{p} \cdot \mathbf{a}} (T(0, \mathbf{a}) \Psi_\lambda) [\phi]. \quad (4.4)$$

These wave functionals can be given a plane-wave normalization

$$\langle \Psi_{\lambda, \mathbf{p}'} | \Psi_{\lambda, \mathbf{p}} \rangle = \delta(\mathbf{p}' - \mathbf{p}). \quad (4.5)$$

The simultaneous eigenstates of mass and linear momentum can be further decomposed into eigenstates of spin, and z -component of spin using

$$\Psi_{\lambda, j, \mathbf{p}, \mu}[\phi] = \sum_{\nu=-j}^j \int_{S(U2)} dR (T(R, 0) \Psi_{\lambda, R^{-1}\mathbf{p}}) [\phi] D_{\mu\nu}^{j*}(R) \quad (4.6)$$

where the integral is over $SU(2)$ and dR is the $SU(2)$ Haar measure. This projection gives the canonical spin. This is the spin measured in the rest frame when the particle is transformed to the rest frame by a rotationless Lorentz transformation. Different projections can be used to get states of different helicities or light-front spins. The integral in equation (4.6) vanishes if there are no states of mass λ and spin j .

The normalization of the states can be chosen so

$$\langle \Psi_{\lambda, j', \mathbf{p}', \mu'} | \Psi_{\lambda, j, \mathbf{p}, \mu} \rangle = \delta(\mathbf{p}' - \mathbf{p}) \delta_{j'j} \delta_{\mu'\mu}. \quad (4.7)$$

The state $|\Psi_{\lambda, j, \mathbf{p}, \mu}\rangle$ is a single particle state if λ is in the discrete spectrum of M .

Since we started from a linear combination of wave functionals, the single-particle state is formally represented by a single-particle wave functional

$$\Psi_{\lambda, j', \mathbf{p}', \mu'}[\phi]. \quad (4.8)$$

In general it is not trivial to compute finite Poincaré transformations in terms of the generators, however if the one-particle state is a non-degenerate state (i.e. the theory has no other particles with the same mass and spin) then this state necessarily transforms irreducibly with respect to the dynamical unitary representation of the Poincaré group. It follows that

$$\begin{aligned} \langle B | U(\Lambda, a) | \Psi_{\lambda, j, \mathbf{p}, \mu} \rangle = \\ \langle B | \psi_{\lambda, \mathbf{p}', \mu'} \rangle \sqrt{\frac{\omega_\lambda(\mathbf{p}')}{\omega_\lambda(\mathbf{p})}} e^{-i\omega_\lambda(\mathbf{p}')a^0 + i\mathbf{p}' \cdot \mathbf{a}} D_{\mu'\mu}^j [\Lambda_c^{-1}(\frac{\mathbf{p}'}{\lambda}) \Lambda_c(\frac{\mathbf{p}}{\lambda})] \end{aligned} \quad (4.9)$$

where

$$(\mathbf{p}')^j = \Lambda^j_0 \omega_\lambda(\mathbf{p}) + \Lambda^j_k \mathbf{p}^k \quad \omega_\lambda(\mathbf{p}) = \sqrt{\lambda^2 + \mathbf{p}^2} \quad (4.10)$$

and $\Lambda_c(\frac{\mathbf{p}}{\lambda})$ is a rotationless Lorentz boost that transforms a frame where a particle of mass λ is at rest to one where it has linear momentum \mathbf{p} .

Normalizable single-particle states have the form

$$\Psi_{\lambda,j,f}[\phi] = \int d\mathbf{p} \sum_{\mu=-j}^j f(\mu, \mathbf{p}) \Psi_{\lambda,j',\mathbf{p}',\mu'}[\phi] \quad (4.11)$$

$$\langle \Psi_{\lambda,j,g} | \Psi_{\lambda,j,f} \rangle = \int d\mathbf{p} \sum_{\mu=-j}^j g^*(\mu, \mathbf{p}) f(\mu, \mathbf{p}). \quad (4.12)$$

In this formalism there is no distinction between elementary particles and bound states. They describe particles because they have discrete mass eigenvalues.

What is interesting is that it is possible to construct the Hilbert space, Poincaré generators, find single-particle states and perform finite Poincaré transformations on single-particle states using only the Euclidean generating functional, and positive-time test functions without performing any analytic continuation.

V. SCATTERING THEORY

In a quantum theory scattering states are solutions of the Schrödinger equation that evolve into asymptotically separated non-interacting single-particle states or bound states. In quantum field theory there is no free dynamics on the physical Hilbert space, so with the exception of one-particle states, there are no states of non-interacting particles on the physical Hilbert space. However, because of cluster properties, there are states that look like states of asymptotically separated particles. These states evolve like systems of free particles until the particles get close enough to interact.

One natural framework to formulate scattering asymptotic conditions that is applicable in both quantum mechanics and quantum field theory is the two-Hilbert space formulation of scattering [10]. In this framework a separate many-particle Hilbert space of non-interacting particles is introduced. This space is used to label the states of the asymptotically stable particles. There is a mapping from this asymptotic space to the physical Hilbert space that adds the correct description of the internal structure of the particles on the physical Hilbert space when the particles are asymptotically separated.

In the asymptotic Hilbert space composite particles are treated like elementary particles with a given mass, spin, and momentum distribution. The internal structure of the composite particle is contained in the mapping to the physical Hilbert space. In field theories all particles have internal structure due to their self-interactions. In the non-relativistic case the mapping from the asymptotic Hilbert space to the physical Hilbert space has the form

$$\prod_i |(\lambda_i, j_i) \mathbf{p}_i, \mu_i\rangle \quad (5.1)$$

where the product is a symmetrized tensor product of possibly composite particle states with given momentum and spin. Normalizable states in the physical Hilbert space are obtained when this mapping is integrated over square integrable functions of the momenta and magnetic quantum numbers of each asymptotically stable particle. In this way composite particles are treated as elementary particles with mass λ_i and spin j_i in the asymptotic Hilbert space while the mapping adds the composite structure of the asymptotically separated bound states in the physical Hilbert space.

In quantum field theory there are two standard approaches to scattering. The most common is the LSZ treatment of scattering, which formulates the scattering asymptotic conditions using interpolating fields that create states from the vacuum that have the same quantum numbers as single particle states. It has the advantage that the asymptotic conditions can be formulated without solving the one-body problem. The price for this advantage is that weak limits must be used to calculate scattering matrix elements. The second approach is Haag-Ruelle scattering [11][12] which uses non-local fields that create only one-particle states from the vacuum. Haag-Ruelle scattering is the natural generalization of standard quantum mechanical scattering in the field theory setting and it has a natural two-Hilbert space formulation[13][14]. Haag-Ruelle scattering states are defined by norm limits, just like in the non-relativistic case. Haag-Ruelle scattering is not commonly used in applications because it requires the solution of the one-body problem on the physical Hilbert space as input. In this work the one-body solutions discussed in the previous section are used to formulate the Haag-Ruelle asymptotic conditions.

We begin with a heuristic summary of the two-Hilbert space formulation of Haag-Ruelle scattering in Minkowski field theory. For simplicity we consider a scalar field theory with a single-particle state of mass λ . To construct the mapping from the asymptotic Hilbert space to the physical Hilbert space the Fourier transform of the field, $\phi(p)$, is

multiplied by a smooth function $\rho(p^2)$ of the square of the Minkowski four momentum that is one when $p^2 = -\lambda^2$ (the mass of the asymptotic particle) and identically vanishes when $-p^2$ is in the rest of spectrum of M^2 .

The product of the Fourier transform of the field and the test function, $\tilde{\phi}_\rho(p) := \rho(p^2)\tilde{\phi}(p)$, is then Fourier transformed back to configuration space. The resulting field $\phi_\rho(x)$ is covariant, but no longer local. It has the property that when it is applied to the physical vacuum it creates only the single-particle eigenstate of the mass operator with mass λ . Because of the multiplication by $\rho(p^2)$, $\phi_\rho(x)$ is a well-behaved operator-valued function of time when it is smeared over a test function in three space variables.

The part of $\phi_\rho(x)$ that asymptotically looks like a creation operator is extracted by taking the linear combination of ϕ and $\dot{\phi}$ below:

$$A(f, t) := -i \int \phi_\rho(x) \overset{\leftrightarrow}{\partial}_0 f(x) d\mathbf{x} \quad (5.2)$$

where $f(x)$ is a smooth positive-energy solution of the Klein-Gordon equation. Here smooth means that the Fourier transform of the $t = 0$ solution is a smooth function with compact support in the three momentum. Haag and Ruelle show that the N -particle scattering states in the physical Hilbert space exist and are given by the strong limits

$$|\Psi_\pm(f_1, \dots, f_N)\rangle = \lim_{t \rightarrow \pm\infty} A(f_N, t) \cdots A(f_1, t) |0\rangle. \quad (5.3)$$

Next we express the limit (5.3) in a two Hilbert space framework that can also be used in the wave functional representation of the physical Hilbert space. First we write $A(f, t)$ defined in (5.2) by expressing the time derivative of the field using the commutator with the Hamiltonian and the time derivative of the Klein-Gordon solution by an energy factor:

$$A(f, t) |0\rangle = -\frac{i}{(2\pi)^{3/2}} \int d\mathbf{x} d\mathbf{p} e^{iHt} (i[H, \phi_\rho(0, \mathbf{x})] - i\omega_\lambda(\mathbf{p})\phi_\rho(0, \mathbf{x})) e^{-iHt} |0\rangle e^{-i\omega_\lambda(\mathbf{p})t + i\mathbf{p}\cdot\mathbf{x}} \tilde{f}(\mathbf{p}) \quad (5.4)$$

where $\tilde{f}(\mathbf{p})$ is a test function in the three momentum. Integrating over \mathbf{x} gives a partial Fourier transform of the field so (5.2) becomes

$$A(f, t) |0\rangle = e^{iHt} \int ([H, \phi_\rho(0, \mathbf{p})] - \omega_\lambda(\mathbf{p})\phi_\rho(0, \mathbf{p})) |0\rangle \tilde{f}(\mathbf{p}) e^{-i\omega_\lambda(\mathbf{p})t} d\mathbf{p}. \quad (5.5)$$

The time-dependence is in the quantities e^{iHt} and $e^{-i\omega_\lambda(\mathbf{p})t}$, where the second factor gives the time dependence of the positive-energy solution the Klein-Gordon equation. This is expressed as an operator that acts on the wave packet $\tilde{f}(\mathbf{p})$ of a free particle of mass λ . It follows that (5.2) can be interpreted as a mapping from a dense subset of the Hilbert space of square integrable functions, $f(\mathbf{p})$, to the physical Hilbert space

$$A(f, t) |0\rangle = e^{iHt} A_1 e^{-iH_0 t} |f\rangle \quad (5.6)$$

where $H_0 = \omega_\lambda(\mathbf{p})$ is the energy of the asymptotic particle and

$$A_1(\mathbf{p}) := ([H, \phi_\rho(0, \mathbf{p})] - \omega_\lambda(\mathbf{p})\phi_\rho(0, \mathbf{p})). \quad (5.7)$$

By repeating this analysis N -times the products that appear in the Haag-Ruelle formula,

$$\prod A(f_1, t) \cdots A(f_N, t) |0\rangle, \quad (5.8)$$

can be expressed as mappings from an N -particle subspace of the Fock space of non-interacting particles of mass λ to the physical Hilbert space:

$$\prod A(f_1, t) \cdots A(f_N, t) |0\rangle = e^{iHt} \int A_N(\mathbf{p}_1, \dots, \mathbf{p}_N) e^{-i(\sum \omega_\lambda(\mathbf{p}_i))t} f_1(\mathbf{p}_1) \cdots f_N(\mathbf{p}_N) d\mathbf{p}_1 \cdots d\mathbf{p}_N = \quad (5.9)$$

$$e^{iHt} \int A_N(\mathbf{p}_1, \dots, \mathbf{p}_N) e^{-iH_0 t} f_1(\mathbf{p}_1) \cdots f_N(\mathbf{p}_N) d\mathbf{p}_1 \cdots d\mathbf{p}_N \quad (5.10)$$

where

$$A_N(\mathbf{p}_1, \dots, \mathbf{p}_N) := \prod ([H, \phi_\rho(0, \mathbf{p}_i)] - \omega_\lambda(\mathbf{p}_i)\phi_\rho(0, \mathbf{p}_i)) |0\rangle. \quad (5.11)$$

Equation (5.9) has the form

$$e^{iHt} A_N e^{-iH_0 t} |\mathbf{f}\rangle \quad (5.12)$$

where $H_0 = \sum \omega_\lambda(\mathbf{p}_i)$ is the Hamiltonian for N non-interacting particles of mass λ .

In this notation the Haag-Ruelle theorem, (5.3) has the two-Hilbert space form:

$$\lim_{t \rightarrow \pm\infty} \| |\Psi_\pm(\mathbf{f})\rangle - e^{iHt} A_N e^{-iH_0 t} |\mathbf{f}\rangle \| = 0. \quad (5.13)$$

Following what is done in standard quantum mechanical multichannel scattering theory, wave operators are defined by

$$\Omega_{N\pm} |\mathbf{f}\rangle := \lim_{t \rightarrow \pm\infty} e^{iHt} A_N e^{-iH_0 t} |\mathbf{f}\rangle = |\Psi_\pm(\mathbf{f})\rangle. \quad (5.14)$$

In the field theory case [12] these wave operators satisfy relativistic intertwining relations

$$U(\Lambda, a) \Omega_{N\pm} = \Omega_{N\pm} (\otimes U_i(\Lambda, a)) \quad (5.15)$$

that relate the dynamical representation of the Poincaré group with the tensor product of n single-particle irreducible representations on the n -particle sector of the asymptotic Fock space.

The scattering states can be expressed using the representation of the physical Hilbert space in terms of wave functionals in section 2. The relevant observation is that

$$\tilde{\phi}_\rho(p) |0\rangle \quad (5.16)$$

is a single-particle state of linear momentum \mathbf{p} and mass λ . The wave functionals (2.7) of section 2 are vectors in the physical Hilbert space, even though they are expressed in terms of Euclidean test functions and the Euclidean generating functional. The wave functional,

$$\Psi_{\lambda, j', \mathbf{p}', \mu'} [\phi], \quad (5.17)$$

defined in the previous section, creates a single-particle state of linear momentum \mathbf{p} and mass λ .

Thus, if we make the replacements

$$\tilde{\phi}_\rho(p) \rightarrow \Psi_{\lambda, j', \mathbf{p}', \mu'} [\phi], \quad (5.18)$$

in the two-Hilbert space Haag-Ruelle injection operator, A_N (5.11), then it becomes the wave functional

$$A_N(\mathbf{p}_1, \mu_1, \dots, \mathbf{p}_N, \mu_N) [\phi] := \prod ([H, \Psi_{\lambda_i, j'_i, \mathbf{p}'_i, \mu'_i}] - \omega_{\lambda_i}(\mathbf{p}_i) \Psi_{\lambda_i, j'_i, \mathbf{p}'_i, \mu'_i}) [\phi] \quad (5.19)$$

where this functional allows for the possibility of scattering of composite particles with higher spin.

This leads to the following formal expression for S -matrix elements between normalizable states:

$$\begin{aligned} S_{fi} &= \langle \Psi_+ | \Psi_- \rangle = \\ &= \lim_{t \rightarrow \infty} \int f_1^*(\mathbf{p}_1, \mu_1) \cdots f_M^*(\mathbf{p}_M, \mu_M) e^{i \sum \omega_{\lambda_i}(\mathbf{p}_i) t} \times \\ &\times \langle A_M^\dagger(\mathbf{p}_1, \mu_1, \dots, \mathbf{p}_M, \mu_M) | e^{-2iHt} | A_N(\mathbf{p}'_1, \mu'_1, \dots, \mathbf{p}'_N, \mu'_N) \rangle \times \\ &\times e^{i \sum \omega_{\lambda_i}(\mathbf{p}'_i) t} f'_1(\mathbf{p}'_1, \mu'_1) \cdots f'_N(\mathbf{p}'_N, \mu'_N) \prod_{ij} d\mathbf{p}_i d\mathbf{p}'_j \end{aligned} \quad (5.20)$$

where the scalar product is expressed in terms of the Euclidean generating functional. This expression has the form

$$S_{fi} = \lim_{t \rightarrow \infty} \langle B(t) | e^{-2iHt} | C(t) \rangle. \quad (5.21)$$

While (5.21) involves operators that are defined in the wave functional representation, the real time evolution operator, e^{-2iHt} , is difficult to calculate in this representation.

Fortunately this quantity can be replaced by a more easily computable quantity using the Kato-Birman invariance principle [15] [16][13][14] which identifies the limits

$$\Omega_{\pm} |\mathbf{f}\rangle := \lim_{t \rightarrow \pm\infty} e^{iHt} A_N e^{-iH_0 t} |\mathbf{f}\rangle = \lim_{t \rightarrow \pm\infty} e^{ig(H)t} A_N e^{-ig(H_0)t} |\mathbf{f}\rangle \quad (5.22)$$

for $g(x)$ in a suitable class of admissible functions provided both limits exist. The content of this result is that in the large-time limit the surviving terms correspond to situations where both exponents oscillate in phase, which requires that both the dynamical and asymptotic energies are the same. Replacing H and H_0 by functions of H resp. H_0 does not change this result provided the function is increasing, with suitable smoothness. A useful choice for $g(x)$ that is in the class of admissible functions is

$$g(x) = -e^{-\beta x} \quad \beta > 0. \quad (5.23)$$

For this choice the expressions for the wave operator becomes

$$\Omega_{\pm} |\mathbf{f}\rangle := \lim_{n \rightarrow \pm\infty} e^{-ine^{-\beta H}} A_N e^{ine^{-\beta H_0}} |\mathbf{f}\rangle \quad (5.24)$$

where the time parameter t has been replaced by a dimensionless integer n .

This means that the expression (5.21) for the S matrix elements can be replaced by

$$\begin{aligned} S_{fi} = \langle \Psi_+ | \Psi_- \rangle &= \lim_{n \rightarrow \infty} \int f_1^*(\mathbf{p}_1, \mu_1) \cdots f_k^*(\mathbf{p}_M, \mu_M) e^{-ine^{-\beta(\sum \omega_{\lambda_i}(\mathbf{p}_i))}} \times \\ &\times \langle A_M^\dagger(\mathbf{p}_1, \mu_1, \dots, \mathbf{p}_M, \mu_M) | e^{2ine^{-\beta H}} | A_N(\mathbf{p}'_1, \mu'_1, \dots, \mathbf{p}'_N, \mu'_N) \rangle \\ &\times e^{-ine^{i\sum \omega_{\lambda_i}(\mathbf{p}'_i)}} f'_1(\mathbf{p}'_1, \mu'_1) \cdots f'_M(\mathbf{p}'_M, \mu'_M) \prod_{ij} d\mathbf{p}_i d\mathbf{p}'_j. \end{aligned} \quad (5.25)$$

The virtue of this expression is that for large *fixed* n , e^{2inx} can be *uniformly* approximated by a polynomial for $x \in [0, 1]$

$$|e^{2inx} - P_{n,\epsilon}(x)| < \epsilon \quad \forall x \in [0, 1]. \quad (5.26)$$

Because the spectrum of $e^{-\beta H}$ is in the interval $[0, 1]$ and the approximation is uniform the operator satisfies the same inequality

$$||| e^{2ine^{-\beta H}} - P_{n,\epsilon}(e^{-\beta H}) ||| < \epsilon \quad (5.27)$$

where the norm on the left is the uniform or operator norm and $P_{n,\epsilon}(x)$ and ϵ are the polynomial and error that appear in equation (5.26).

Compared to (5.21) equation (5.25) has the form

$$S_{fi} = \lim_{n \rightarrow \infty} \sum_m d_m(n) \langle B(n) | e^{-\beta m H} | C(n) \rangle. \quad (5.28)$$

where $d_m(n)$ are the coefficients of the polynomial in (5.27). This is useful because matrix elements of powers of $e^{-m\beta H}$ between wave functionals $B(n)[\phi]$ and $C(n)[\phi]$ can be expressed directly in terms of the generating functional using (3.18).

While time-dependent methods are not traditionally used in scattering calculations, they have been used successfully in non-relativistic few-body calculations [17]. The advantage of the above formalism is that the entire calculation can be performed using only Euclidean methods.

In order to calculate sharp-momentum transition matrix elements it is necessary to use narrow wave packets. If the transition matrix is sufficiently smooth as a function of momentum, it will factor out of the S -matrix element, allowing one to define scattering observables that do not depend on the details of the wave packet. This is an assumption in the standard formulation relating time-dependent and time-independent scattering [18]. For sharp initial and final wave packets the on-shell transition matrix elements can be approximated by:

$$\begin{aligned} & \langle \mathbf{p}_1, \mu_1, \dots \mathbf{p}_N, \mu_N | T | \mathbf{p}_1, \mu_1, \dots \mathbf{p}_2, \mu_2 \rangle \\ & \approx \frac{\langle B | S | C \rangle - \langle B | C \rangle}{-2\pi i \langle B | \delta^4(p_f - p_i) | C \rangle}. \end{aligned} \quad (5.29)$$

After the wave packets are fixed the limit $n \rightarrow \infty$ in (5.25) can be investigated. For a large enough n the term in the limit, (5.25), which has the form

$$\langle B(n) | e^{-2ine^{-\beta H}} | C(n) \rangle, \quad (5.30)$$

will be a good approximation to

$$\langle B | S | C \rangle = \langle B(0) | S | C(0) \rangle. \quad (5.31)$$

For this value of n , $e^{-2ine^{-\beta H}}$ can then be uniformly approximated by a polynomial in $e^{-\beta H}$ which can be evaluated using Euclidean methods.

$$\langle B(n) | e^{-2ine^{-\beta H}} | C(n) \rangle \approx \sum_m d_m(n) \langle B(n) | e^{-m\beta H} | C(n) \rangle. \quad (5.32)$$

Combining these three approximations gives an approximation to sharp-momentum transition matrix elements using matrix elements of $e^{-n\beta H}$ in normalizable states as input.

Once the scattering states are known their Poincaré transformation properties are determined by equation (5.15) and the transformation properties (4.9) of the single-particle states.

It is useful summarize the steps needed to calculate transition-matrix elements.

1. Solve the one-body problem. These are eigenstates of the mass-square operator with discrete eigenvalues: $\Phi[\phi]$.
2. Use translational and rotational covariance to construct $\Phi((\lambda, j), \mathbf{p}, \mu)[\phi]$.
3. Choose a sufficiently narrow set of single asymptotic particle wave packets $f(\mathbf{p}_i, \mu_i)$. The width must be sufficiently narrow to factor the transition matrix elements out of the S -matrix elements.
4. Construct the two-Hilbert space mappings

$$A_n(\mathbf{f}) := \prod_i \int ([H, \Phi((\lambda, j)\mathbf{p}_i, \mu_i)] - \omega_\lambda(\mathbf{p}_i) \Phi((\lambda, j)\mathbf{p}_i, \mu_i)) f(\mathbf{p}_i, \mu_i) d\mathbf{p}_i e^{ine^{-\beta(\sum \omega_\lambda(\mathbf{p}_i))}} \quad (5.33)$$

5. Pick a large enough n .

6. Make a polynomial approximation to e^{2inx} for $x \in [0, 1]$

$$e^{2inx} \approx P_{2n, \epsilon}(x). \quad (5.34)$$

7. Calculate

$$S_{fi} = \langle A(\mathbf{f}', n) | P_{2n, \epsilon}(e^{-\beta H}) | A(\mathbf{f}', n) \rangle \quad (5.35)$$

8. Approximate $\langle \mathbf{p}_1, \mu_1, \dots \mathbf{p}_n, \mu_n | T | \mathbf{p}_1, \mu_1, \dots \mathbf{p}_2, \mu_2 \rangle$ using (5.29).

The result can be expressed directly in terms of the Euclidean generating functional using (3.18).

The discussion above assumes that the starting point is a Euclidean generating functional. For models the requirement that all of these approximations converge are model assumptions that restrict properties of the model generating functional or Green functions. These are reasonable requirements, since the model Green functions are modeled after the field theoretic Green functions, which are expected to have these properties, at least approximately.

The usual difficulties of realizing the Poincaré symmetry are replaced by the requirement of finding reflection positive Euclidean invariant Green functions or generating functionals. It is interesting that the scattering matrix constructed above will be unitary, even if the Green function is given perturbatively.

The proof of the existence of the wave operators in the Haag-Ruelle formulation of scattering depends on properties of the field theory that may be violated on approximation. For the case of models the existence of channel wave operators can be established by a generalization of Cook's method[19], which gives the following sufficient condition for the existence of N-particle wave operators:

$$\int_0^{\pm\infty} \|([H, A_N] - A_N H_0) e^{-iH_0 t} |\mathbf{f}\rangle\| dt < \infty. \quad (5.36)$$

For the simple kinds of two-body models discussed in section VII it is easy to see that the generalized Cook condition (5.36) translates into regularity properties of the connected four-point function. This is because

$$\|([H, A_2] - A_2 H_0) e^{-iH_0 t} |\mathbf{f}\rangle\|^2 \quad (5.37)$$

is linear in the four-point Euclidean Green function and vanishes when the Green function is replaced by the free four-point Euclidean Green function. What controls the integral in (5.37) is the difference between the full and free four-point Euclidean Green functions, which is the connected Euclidean four-point function.

Finally we note that even though the calculation of the scattering observables are based on Euclidean quantities, the mechanism that leads to the convergence of the wave operators is in-phase oscillations, not an exponential fall-off. The scattering states involve strong limits and replace interpolating fields by fields that create single-particle states out of the vacuum. As a result, these calculations are not subject to some of the difficulties encountered in scattering calculations based on a Euclidean lattice discretization[20], however it is necessary to be able to accurately compute matrix elements of $e^{-\beta H}$.

VI. GREEN FUNCTION REPRESENTATION

While the Euclidean generating functionals provide a concise and elegant description of the theory as well as a consistent treatment of the few and many-body problems, the direct Green function approach of Osterwalder and Schrader may be more appropriate for constructing phenomenological few-body models.

In the Green function approach the representation of the physical Hilbert space in terms of Euclidean wave functionals is replaced by sequences of positive-time support functions of the form:

$$B[\phi] \rightarrow \langle \mathbf{x} | \mathbf{f} \rangle := \begin{pmatrix} f_0 \\ f_1(\mathbf{x}_{11}) \\ f_2(\mathbf{x}_{21}, \mathbf{x}_{22}) \\ \vdots \end{pmatrix} \quad (6.1)$$

with an inner product that is expressed in terms of multipoint Euclidean Green functions.

$$\langle \mathbf{g} | \mathbf{f} \rangle := \sum_{mn} \int dx_1 \cdots dx_m dy_n \cdots dy_1 g_m^*(\Theta x_1 \cdots \Theta x_m) G_{m+n}(x_1, \cdots, x_m, y_n, \cdots, y_1) f_n(y_1, \cdots, y_n). \quad (6.2)$$

In the Green function representation the support of $f_n(y_1, \cdots, y_n)$ is for $0 < y_1^0 < y_2^0 < \cdots$. Note that the order of the support of the Euclidean times is identical to the order of the fields in the corresponding Minkowski Wightman function. Reflection positivity is the condition

$$\langle \mathbf{f} | \mathbf{f} \rangle \geq 0 \quad (6.3)$$

and $\langle \mathbf{g} | \mathbf{f} \rangle$ is the physical quantum mechanical scalar product.

The relation between the Euclidean and Minkowski scalar products is illustrated for the case of a free Euclidean two point function, one for spin zero and one for spin 1/2:

$$\begin{aligned}
\langle \mathbf{f} | \mathbf{f} \rangle &= \int f(x) G_2(\Theta x, y) f(y) d^4 x d^4 y = \\
&= \frac{1}{(2\pi)^4} \int d^4 x d^4 y d^4 \mathbf{p} f(\mathbf{x}) \frac{e^{i\mathbf{p} \cdot (\theta \mathbf{x} - y)}}{\mathbf{p}^2 + m^2} f(y) \\
&= \frac{1}{(2\pi)^4} \int d^4 x d^4 y d^4 \mathbf{p} f(\mathbf{x}) \frac{e^{-i\mathbf{p}_0 \cdot (\mathbf{x}_0 + y_0) + i\vec{\mathbf{p}} \cdot (\vec{\mathbf{x}} - \vec{\mathbf{y}})}}{(\mathbf{p}^0 + i\omega_m(\vec{\mathbf{p}}))(\mathbf{p}^0 - i\omega_m(\vec{\mathbf{p}}))} f(y) \\
&= \int d^3 \mathbf{p} \frac{|g(\vec{\mathbf{p}})|^2}{2\omega_m(\vec{\mathbf{p}})} \geq 0
\end{aligned} \tag{6.4}$$

where

$$g(\vec{\mathbf{p}}) := \frac{1}{(2\pi)^{3/2}} \int d^4 y f(y) e^{-\omega_m(\vec{\mathbf{p}}) y_0 - i\vec{\mathbf{p}} \cdot \vec{\mathbf{y}}}. \tag{6.5}$$

For spin 1/2:

$$\begin{aligned}
\langle \mathbf{f} | \mathbf{f} \rangle &= \int f(x) \gamma^0 G_2(\Theta x, y) f(y) d^4 x d^4 y = \\
&= \frac{1}{(2\pi)^4} \int d^4 x d^4 y d^4 \mathbf{p} f(\mathbf{x}) e^{i\mathbf{p} \cdot (\theta \mathbf{x} - y)} \gamma^0 \frac{m - \mathbf{p} \cdot \boldsymbol{\gamma}_e}{\mathbf{p}^2 + m^2} f(y) \\
&= \int g^\dagger(\vec{\mathbf{p}}) \frac{\Lambda_+(p)}{(2\pi)^3} g(\vec{\mathbf{p}}) d^3 \mathbf{p}
\end{aligned} \tag{6.6}$$

where

$$\Lambda_+(p) := \frac{\omega_m(\vec{p}) + \gamma^0 \vec{\gamma} \cdot \vec{p} - m \gamma^0}{2\omega_m(\vec{p})}. \tag{6.7}$$

We see clearly that in both case the Euclidean integrals with the Euclidean time reversal are identical to the corresponding Minkowski scalar products. The equality shows that analytic continuation is not required to compute the physical scalar product in the Euclidean representation.

In the Green-function representation the formulas for the Poincaré generators are replaced by

$$\langle \mathbf{x} | H | \mathbf{f} \rangle := \{0, \frac{\partial}{\partial \mathbf{x}_{11}^0} f_1(\mathbf{x}_{11}), \left(\frac{\partial}{\partial \mathbf{x}_{21}^0} + \frac{\partial}{\partial \mathbf{x}_{22}^0} \right) f_2(\mathbf{x}_{21}, \mathbf{x}_{22}), \dots\} \tag{6.8}$$

$$\langle \mathbf{x} | \mathbf{P} | \mathbf{f} \rangle := \{0, -i \frac{\partial}{\partial \vec{\mathbf{x}}_{11}} f_1(\mathbf{x}_{11}), -i \left(\frac{\partial}{\partial \vec{\mathbf{x}}_{21}} + \frac{\partial}{\partial \vec{\mathbf{x}}_{22}} \right) f_2(\mathbf{x}_{21}, \mathbf{x}_{22}), \dots\} \tag{6.9}$$

$$\langle \mathbf{x} | \mathbf{J} | \mathbf{f} \rangle := \{0, -i \vec{\mathbf{x}}_{11} \times \frac{\partial}{\partial \vec{\mathbf{x}}_{11}} f_1(\mathbf{x}_{11}), -i \left(\vec{\mathbf{x}}_{21} \times \frac{\partial}{\partial \vec{\mathbf{x}}_{21}} + \vec{\mathbf{x}}_{22} \times \frac{\partial}{\partial \vec{\mathbf{x}}_{22}} \right) f_2(\mathbf{x}_{21}, \mathbf{x}_{22}), \dots\} \tag{6.10}$$

$$\langle \mathbf{x} | \mathbf{K} | \mathbf{f} \rangle := \{0, \left(\vec{\mathbf{x}}_{11} \frac{\partial}{\partial \mathbf{x}_{11}^0} - \mathbf{x}_{11}^0 \frac{\partial}{\partial \vec{\mathbf{x}}_{11}} \right) f_1(\mathbf{x}_{11}),$$

$$\left(\vec{x}_{21} \frac{\partial}{\partial \mathbf{x}_{21}^0} - \mathbf{x}_{21}^0 \frac{\partial}{\partial \vec{x}_{21}} + \vec{x}_{22} \frac{\partial}{\partial \mathbf{x}_{22}^0} - \mathbf{x}_{22}^0 \frac{\partial}{\partial \vec{x}_{22}} \right) f_2(\mathbf{x}_{21}, \mathbf{x}_{22}), \dots \}. \quad (6.11)$$

For particles with spin these expressions are modified as follows

$$\mathbf{J}: \quad \left(-i\vec{x}_{11} \times \frac{\partial}{\partial \vec{x}_{11}} \right) \rightarrow \left(-i\vec{x}_{11} \times \frac{\partial}{\partial \vec{x}_{11}} + \vec{\Sigma} \right) \quad (6.12)$$

$$\mathbf{K}: \quad \left(\vec{x}_{11} \frac{\partial}{\partial \mathbf{x}_{11}^0} - \mathbf{x}_{11}^0 \frac{\partial}{\partial \vec{x}_{11}} \right) \rightarrow \left(\vec{x}_{11} \frac{\partial}{\partial \mathbf{x}_{11}^0} - \mathbf{x}_{11}^0 \frac{\partial}{\partial \vec{x}_{11}} + \vec{B} \right) \quad (6.13)$$

where

$$\vec{\Sigma} = i\vec{\nabla}_\phi S(e^{\frac{-i}{2}\vec{\sigma}\cdot\vec{\phi}}, e^{\frac{i}{2}\vec{\sigma}^t\cdot\vec{\phi}})_{aa'} \quad (6.14)$$

and

$$\vec{B} = \vec{\nabla}_\rho S(e^{\frac{-i}{2}\vec{\sigma}\cdot\vec{\rho}}, e^{\frac{-i}{2}\vec{\sigma}^t\cdot\vec{\rho}})_{aa'} \quad (6.15)$$

Here $S(R_1, R_2)$ is a finite dimensional representation of $SU(2) \times SU(2)$ associated with the type of field. It is constructed by expressing the finite dimensional representation of the Lorentz group $S(\Lambda)$ in terms of $SL(2, C)$ matrices $S(A, A^*)$ and subsequently replacing A and A^* by independent unitary matrices, A and B .

For the case of fermions the Euclidean time reversal operator also includes a factor γ^0 for each final particle.

The formulas summarized in this section are discussed in more detail in [21].

VII. FEW-BODY MODELS

A typical application where relativistic few-body methods are used is elastic nucleon-nucleon scattering. This is normally treated using the inhomogeneous Bethe-Salpeter equation. We outline the formulation of this problem in the Euclidean quantum mechanical representation.

We consider a model Green function of the form

$$\Theta S \rightarrow \begin{pmatrix} 0 & 0 & 0 & \dots\dots \\ 0 & S_2(\Theta \mathbf{x}_{11}, \mathbf{x}_{12}) & 0 & \\ 0 & 0 & S_4(\Theta \mathbf{x}_{21}, \Theta \mathbf{x}_{22}, \mathbf{x}_{23}, \mathbf{x}_{24}) & 0 \\ \vdots & 0 & 0 & \ddots \end{pmatrix}. \quad (7.1)$$

For this model we assume that only S_2 and S_4 are non-zero. Furthermore we assume that S_2 and S_4 are related by cluster properties:

$$S_4 = S_2 S_2 + S_c = S_0 + S_c. \quad (7.2)$$

The Euclidean Bethe Salpeter kernel is defined by

$$S_4^{-1} - S_0^{-1} = -K. \quad (7.3)$$

The structure of S_2 is determined by covariance up to an unknown Lehmann weight. If the weight is a delta function in the mass then this is a free field Euclidean Green function. In this case the one-body solutions that are needed to formulate the scattering problem are trivial. If the Lehmann weight also includes some continuous spectrum then it is necessary to solve a one-body problem to formulate the scattering asymptotic condition. To do this we take an orthonormal set of positive-time test functions and use the Gram-Schmidt method to construct an orthonormal set

$$\langle f_n | f_m \rangle = \int dx dy f_n^*(\Theta \mathbf{x}) \gamma^0 S_2(\mathbf{x} - \mathbf{y}) f_m(\mathbf{y}) = \delta_{mn}. \quad (7.4)$$

Because the invariant Minkowski Green function is defined with a Dirac conjugate field rather than a Hilbert space adjoint, the γ^0 needs to be eliminated from S_2 to get the continuation to the Wightman function that serves as the kernel of the Hilbert space scalar product. This is achieved by including γ^0 as the spinor part of the Θ operator:

In this basis one-body solutions have the form

$$|\lambda\rangle = \sum_n c_n |\mathbf{f}_n\rangle \quad (7.5)$$

where c_n and λ are determined solving the eigenvalue problem for discrete λ^2 :

$$\lambda^2 c_n = \sum_m \left(\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \mathbf{a}^2} \right) (f_n \gamma^0 \Theta S_2 f_{m,I,(\tau,\mathbf{a})})_{\tau=\mathbf{a}=0} c_m. \quad (7.6)$$

In this case the Euclidean two-point Green function has the form

$$S_2(\mathbf{x} - \mathbf{y}) := \frac{1}{(2\pi)^4} \int d^4 \mathbf{p} \rho(m) dm \frac{m - \mathbf{p} \cdot \boldsymbol{\gamma}_e}{\mathbf{p}^2 + m^2} e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} \quad (7.7)$$

where

$$i\gamma_{0e} = \beta = \gamma^0 = -\gamma_0; \quad \gamma_e^i = \gamma^i. \quad (7.8)$$

and

$$\rho(m) = \rho_m \delta(m - \lambda) + \rho_c(m). \quad (7.9)$$

The matrix elements have the form

$$\begin{aligned} & (f, \Theta \gamma^0 S_2 f) \\ &= \frac{1}{(2\pi)^4} \int d^4 \mathbf{x} d^4 \mathbf{y} d^4 \mathbf{p} dm f(\mathbf{x}) e^{i\mathbf{p} \cdot (\Theta \mathbf{x} - \mathbf{y})} \gamma^0 \frac{m - \mathbf{p} \cdot \boldsymbol{\gamma}_e}{\mathbf{p}^2 + m^2} \rho(m) f(\mathbf{y}) \\ &= \int g^\dagger(\mathbf{p}, m) \frac{\Lambda_+(\mathbf{p}, m)}{(2\pi)^3} \rho(m) g(\vec{\mathbf{p}}, m) d\mathbf{p} dm \end{aligned} \quad (7.10)$$

where

$$\Lambda_+(p) := \frac{\omega_m(\mathbf{p}) + \gamma^0 \vec{\boldsymbol{\gamma}} \cdot \mathbf{p} - m\gamma^0}{2\omega_m(\mathbf{p})} \quad (7.11)$$

is the positive energy Dirac projector and

$$g(\mathbf{p}, m) := \int d^4 \mathbf{x} e^{-\omega_m(\mathbf{p})\mathbf{x}_0 - i\mathbf{p} \cdot \mathbf{x}} f(\mathbf{x}). \quad (7.12)$$

Single particle eigenstates of mass, linear momentum and spin are constructed from the mass eigenstates (7.6)

$$\psi_\lambda(\mathbf{x}) = \sum_n c_n f_n(\mathbf{x}) \quad (7.13)$$

using

$$\psi_\lambda(\mathbf{p}, \mu) = \sum_n c_n \frac{1}{(2\pi)^{3/2}} \int \sum_\nu \mathbf{f}_n(\tau, R^{-1}\mathbf{x} - \mathbf{a}) e^{-iR^{-1}\mathbf{p} \cdot \mathbf{x}} d\mathbf{a} D_{\mu\nu}^{j*}(R) dR \quad (7.14)$$

The Haag-Ruelle operators $A(\mathbf{p}, \mu)$ are

$$A(\mathbf{p}, \mu)[\mathbf{x}] = \frac{1}{(2\pi)^{3/2}} \int d\mathbf{p} \left[-\frac{\partial}{\partial \beta} - \omega_\lambda(\mathbf{p}) \right] \psi_\lambda(\mathbf{p}, \mu) [\tau - \beta, \mathbf{x} - \vec{\mathbf{a}}]_{|\beta=\mathbf{a}=0} \quad (7.15)$$

The scattering asymptotic states of interest are two-body states. S -matrix elements in normalizable states can be computed using the methods discussed in section 5. Equation (5.29) is replaced by:

$$\langle g|S|f\rangle =$$

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \sum_{d_n} \int g(\mathbf{p}'_1, \mu'_1) g(\mathbf{p}'_2, \mu'_2) e^{i n e^{-\beta(\omega_\lambda(\mathbf{p}'_1) + \omega_\lambda(\mathbf{p}'_2))}} A^\dagger(\mathbf{p}'_1, \mu'_1) [\Theta \mathbf{x}'_2] A^\dagger(\mathbf{p}'_2, \mu'_2) [\Theta \mathbf{x}'_1] \gamma_1^0 \gamma_2^0 \\
& \times S_4(\mathbf{x}'_2, \mathbf{x}'_2; \mathbf{x}_1, \mathbf{x}_2) A(\mathbf{p}_1, \mu_1) [\mathbf{x}_1 - 2n\beta] A(\mathbf{p}_2, \mu_2) [\mathbf{x}_2 - 2n\beta] e^{i n e^{-\beta(\omega_\lambda(\mathbf{p}'_1) + \omega_\lambda(\mathbf{p}'_2))}} \times \\
& g(\mathbf{p}_2, \mu_2) g(\mathbf{p}_1, \mu_1) d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}'_1 d\mathbf{p}'_2 dx_2 dx'_2 dx'_1 dx'_2
\end{aligned} \tag{7.16}$$

Here the reflection positivity is limited to requiring that $\gamma_1^0 \gamma_2^0 \Theta S_4$ is non-negative on products of positive-time test functions.

This illustrates how the approximations discussed above can be implemented in a few-body setting. We note that even if S_4 is calculated perturbatively, the resulting approximate S matrix will be unitary.

VIII. SCATTERING TEST

The scattering computations outlined above and in section V are based on the convergence of a sequence of three approximations that are performed in a prescribed order. While they should in principle converge for suitable model Green functions, that does not imply that the approximations can be sufficiently well-controlled to give converged predictions for reactions at the few GeV energy scale of interest. Since to the best of our knowledge this approach to scattering, i.e. computing sharp-momentum transition matrix elements using matrix elements of $e^{-n\beta H}$ in normalizable states as input, has not even been tested in non-relativistic models, we discuss the implementation of this method in an exactly solvable non-relativistic model. This has the advantage that all of the approximations can be compared to exact results, and the accuracy of the proposed method can be determined precisely. We consider a non-relativistic Hamiltonian with a separable potential that has the range and strength of a nucleon-nucleon interaction. The range is fixed by the pion mass while the strength is adjusted to bind two nucleons with the deuteron binding energy.

The interaction is taken as a Yamaguchi interaction, with Hamiltonian

$$H = \mathbf{k}^2/m - |g\rangle \lambda \langle g| \quad \langle \mathbf{k}|g\rangle = g(\mathbf{k}) = \frac{1}{m_\pi^2 + \mathbf{k}^2}. \tag{8.1}$$

The transition matrix elements are

$$\langle \mathbf{k}|t|\mathbf{k}'\rangle = g(\mathbf{k}) \tau\left(\frac{k'^2}{m_n} + i0^+\right) g(\mathbf{k}') \tag{8.2}$$

with

$$\tau\left(\frac{k'^2}{m_n} + i0^+\right) = -\frac{\lambda}{1 + \frac{m_n \lambda \pi^2}{m_\pi} \frac{1}{(im_\pi + k')^2}} \tag{8.3}$$

where $m_n = .94\text{GeV}$, $m_\pi = .14\text{GeV}$, $e_b = -2.24\text{MeV}$ and the coupling constant is determined from these parameters by

$$\lambda = \frac{m_\pi}{m_n \pi^2} (m_\pi + \sqrt{-m_n e_b})^2. \tag{8.4}$$

To test the approximations we calculate sharp-momentum transition matrix elements using matrix elements of $e^{-\beta H}$ evaluated between normalizable states using the methods outlined in section five. While the eigenstates of this model can be computed exactly, we had to use the spectral expansion of the Hamiltonian to compute matrix elements of $e^{-\beta H}$. While this is a complicated construction in the non-relativistic case, it is replaced by an elementary quadrature (3.18) in the Euclidean case.

The first approximation is to extract sharp-momentum transition matrix elements using sufficiently narrow wave packets in equation (5.29). For this model exact expressions are available for both the S operator and transition matrix. For our test problem we choose Gaussian wave packets in the relative momenta of the form

$$\phi(k) = N e^{-\alpha(k-k_0)^2} = N e^{-(k-k_0)^2/k_w^2} \tag{8.5}$$

where N is a normalization constant and k_0 is the on-shell momentum. We do not choose a particular direction because the interaction is pure s -wave, and the on-shell transition matrix elements at a given energy are given by a single complex number.

Sharp on-shell transition matrix elements computed exactly and approximately from S -matrix elements using (5.29) are compared as a function of the width, k_w of the wave packets. The wave packet widths were determined by the requirement that the approximate transition matrix elements agree with the exact transition matrix elements to an accuracy of less than 0.1%. The results are shown in Table 1 as a function of the relative momentum. The first column of Table 1 shows the on-shell momentum k_0 (center of the Gaussian). The second column shows the value of α used to get the error shown in the fourth column. All of the errors in column four round up to .1%. The third column lists $k_w := 1/\sqrt{\alpha}$ for each value of k_0 and the last column is the dimensionless ratio k_w/k_0 . The values of α in column 3 are used in all of the calculations in this section.

Table 1

k_0	α	k_w	% error	k_w/k_0
[GeV]	[GeV ⁻²]	[GeV]		
0.05	325000	0.00175412	0.1	0.035
0.1	105000	0.00308607	0.1	0.030
0.2	26000	0.00620174	0.1	0.031
0.3	10500	0.009759	0.1	0.032
0.4	5100	0.0140028	0.1	0.035
0.5	3000	0.0182574	0.1	0.036
0.6	2000	0.0223607	0.1	0.037
0.7	1350	0.0272166	0.1	0.038
0.8	1000	0.0316228	0.1	0.039
0.9	750	0.0365148	0.1	0.040
1.0	600	0.0408248	0.1	0.040
1.1	475	0.0458831	0.1	0.041
1.2	400	0.05	0.1	0.041
1.3	330	0.0550482	0.1	0.042
1.4	290	0.058722	0.1	0.041
1.5	250	0.0632456	0.1	0.042
1.6	210	0.0690066	0.1	0.043
1.7	190	0.0725476	0.1	0.042
1.8	170	0.0766965	0.1	0.042
1.9	150	0.0816497	0.1	0.042
2.0	135	0.0860663	0.1	0.043

The last column of Table 1 suggests that a .1% error will generally be obtained if this width is less than 3% of the on-shell momentum. This property holds over a wide range of momenta in this model. This is a simple transition operator so one may anticipate narrower wave packets are needed for more realistic models. This approximation can be improved by decreasing the width of the wave packet; it is the largest source of error in the calculations. In any realistic calculation it does not have to be better than the experimental resolution.

The second step is to approximate S -matrix elements in these Gaussian wave packets using equation (5.25). It is important to first pick the wave packets because the n value needed for convergence depends on the width of the wave packet. The approximate quantities are

$$\langle \phi | S_n | \phi \rangle := \int d\mathbf{k} d\mathbf{k}' \phi(\mathbf{k}) e^{-i n e^{-\beta k^2/m}} \langle \mathbf{k} e | e^{-2i n e^{-\beta H}} | \mathbf{k}' \rangle e^{-i n e^{-\beta k'^2/m}} \phi(\mathbf{k}') \quad (8.6)$$

In Tables 2-8 these quantities are computed using the spectral expansion for H . In these calculations the bound-state contribution is not included because it vanishes in the large n limit. Tables 2-8 show the real and imaginary parts of matrix elements $\langle \phi | (S_n - I) | \phi \rangle$ for different values of n for $k_0 = 50, 100, 200, 500, 1000, 1500, 2000$ MeV. The exact value is given at the bottom of each table. Table 9 shows the values of β , the product $k_0 \times \beta$ and the n -values used in our final calculations. Table 9 suggests that β should be chosen so $k_0 \times \beta$ is of order unity. Except for the $k_0 = 50$ MeV case, $n = 250$ or more gives errors for the n -limits that are smaller than the errors made in the factorization approximation, (5.29).

The n dependence of the real and imaginary part of matrix elements of $\langle \phi | (S - I) | \phi \rangle$, computed using (5.25), are

plotted as a function of n for different values of k_0 in figures 1-10. Figures 11 and 12 show how fast the neglected bound state contribution to the spectral expansion fall off with n for $k_0 = 1$ GeV.

Table 2: $k_0 = 50[\text{MeV}]$, $\alpha = 325000[\text{GeV}^{-2}]$

n	Re $\langle\phi (S_n - I) \phi\rangle$	Im $\langle\phi (S_n - I) \phi\rangle$
50	-7.62976513315350e-1	-1.52406978178214e-1
100	-1.33113144491104e+0	-2.75546806155677e-1
150	-1.67324498184421e+0	-3.50392186517949e-1
200	-1.83391449191883e+0	-3.86136590065981e-1
250	-1.89273779093641e+0	-3.99389077283171e-1
300	-1.90951807884485e+0	-4.03228425149703e-1
350	-1.91324926322936e+0	-4.04093826445785e-1
400	-1.91389545311571e+0	-4.04245969269804e-1
450	-1.91398265491004e+0	-4.04266741575048e-1
500	-1.91399177708580e+0	-4.04268944214980e-1
550	-1.91399249452470e+0	-4.04269131167755e-1
600	-1.91399253056843e+0	-4.04269144047622e-1
650	-1.91399252976074e+0	-4.04269145893613e-1
ex	-1.91399253060872e+0	-4.04269147714400e-1

Table 3: $k_0 = 100[\text{MeV}]$, $\alpha = 105000[\text{GeV}^{-2}]$

n	Re $\langle\phi (S_n - I) \phi\rangle$	Im $\langle\phi (S_n - I) \phi\rangle$
50	-8.73395186664514e-1	4.95616337213744e-1
100	-1.34576615227520e+0	7.59199494502660e-1
150	-1.49091126760062e+0	8.39700869213905e-1
200	-1.51566533604846e+0	8.53352070852317e-1
250	-1.51799902547669e+0	8.54631681615040e-1
300	-1.51811943431498e+0	8.54697554653043e-1
350	-1.51812278309620e+0	8.54699376334219e-1
400	-1.51812288480017e+0	8.54699423334768e-1
450	-1.51812290596551e+0	8.54699435479409e-1
500	-1.51812290955424e+0	8.54699438102000e-1
550	-1.51812290968227e+0	8.54699438639971e-1
600	-1.51812288857871e+0	8.54699427676778e-1
650	-1.51812275938123e+0	8.54699356334623e-1
ex	-1.51812291315971e+0	8.54699438329052e-1

Table 4: $k_0 = 200[\text{MeV}]$, $\alpha = 26000[\text{Gev}^{-2}]$

n	Re $\langle \phi (S_n - I) \phi \rangle$	Im $\langle \phi (S_n - I) \phi \rangle$
50	-2.08408481834932e-1	4.56550768265380e-1
100	-3.11945696198071e-1	6.85279276148059e-1
150	-3.38623394392403e-1	7.44641333032490e-1
200	-3.42127100784575e-1	7.52475859236454e-1
250	-3.42359208499266e-1	7.52997722504423e-1
300	-3.42366821259122e-1	7.53015087494957e-1
350	-3.42366956571344e-1	7.53015382645461e-1
400	-3.42366963936415e-1	7.53015400508140e-1
450	-3.42366965122021e-1	7.53015404697748e-1
500	-3.42366965247355e-1	7.53015405389574e-1
550	-3.42366962667700e-1	7.53015400608940e-1
600	-3.42366938358008e-1	7.53015348812648e-1
650	-3.42366884413759e-1	7.53015247619997e-1
ex	-3.42366967477707e-1	7.53015410457076e-1

Table 5: $k_0 = 500[\text{MeV}]$, $\alpha = 3000[\text{Gev}^{-2}]$

n	Re $\langle \phi (S_n - I) \phi \rangle$	Im $\langle \phi (S_n - I) \phi \rangle$
50	-5.93330385580271e-3	9.65738963854834e-2
100	-7.12681692349637e-3	1.18415504225673e-1
150	-7.18129565909453e-3	1.19491569949089e-1
200	-7.18179706798405e-3	1.19502423190014e-1
250	-7.18179794641838e-3	1.19502444469971e-1
300	-7.18179794670870e-3	1.19502444477783e-1
350	-7.18179794671048e-3	1.19502444477784e-1
400	-7.18179794671081e-3	1.19502444477784e-1
ex	-7.18179797016073e-3	1.19502444795275e-1

Table 6: $k_0 = 1[\text{GeV}]$, $\alpha = 600[\text{Gev}^{-2}]$

n	Re $\langle \phi (S_n - I) \phi \rangle$	Im $\langle \phi (S_n - I) \phi \rangle$
50	-1.47024820732811e-4	1.55922557816223e-2
100	-1.62726188649875e-4	1.79713868930101e-2
150	-1.62967714125273e-4	1.80219591282450e-2
200	-1.62968113934903e-4	1.80220978916775e-2
250	-1.62968113982642e-4	1.80220979403700e-2
300	-1.62968113982642e-4	1.80220979403721e-2
350	-1.62968113982753e-4	1.80220979403720e-2
400	-1.62968113982975e-4	1.80220979403718e-2
ex	-1.62968113982742e-4	1.80220979403858e-2

Table 7: $k_0 = 1.5[\text{GeV}]$, $\alpha = 250[\text{GeV}^{-2}]$

n	Re $\langle \phi (S_n - I) \phi \rangle$	Im $\langle \phi (S_n - I) \phi \rangle$
50	-1.40242356887477e-5	4.66175982621713e-3
100	-1.54430995201738e-5	5.52235009412764e-3
150	-1.54679181726403e-5	5.55112445776045e-3
200	-1.54679281958447e-5	5.55130663695727e-3
250	-1.54679280390813e-5	5.55130688968718e-3
300	-1.54679280390813e-5	5.55130688978369e-3
350	-1.54679280394143e-5	5.55130688978374e-3
400	-1.54679280388592e-5	5.55130688978377e-3
ex	-1.54679280242191e-5	5.55130688386830e-3

Table 8: $k_0 = 2.0[\text{GeV}]$, $\alpha = 135[\text{GeV}^{-2}]$

n	Re $\langle \phi (S_n - I) \phi \rangle$	Im $\langle \phi (S_n - I) \phi \rangle$
50	-2.60094316473225e-6	1.94120750171791e-3
100	-2.82916859895010e-6	2.35553585404449e-3
150	-2.83171624670953e-6	2.37471383801820e-3
200	-2.83165946257657e-6	2.37492460997990e-3
250	-2.83165905312632e-6	2.37492527186858e-3
300	-2.83165905257121e-6	2.37492527262432e-3
350	-2.83165905190508e-6	2.37492527262493e-3
400	-2.83165905234917e-6	2.37492527262540e-3
ex	-2.83165905227843e-6	2.37492527259701e-3

Table 9: Parameters

k_0 [GeV]	β [GeV $^{-1}$]	$k_0 \times \beta$	n
0.05	80.0	4.0	630
0.1	40.0	4.0	450
0.2	10.0	2.0	470
0.3	5.0	1.5	330
0.4	4.0	1.6	235
0.5	3.0	1.5	205
0.6	2.5	1.5	225
0.7	1.6	1.2	200
0.8	1.4	1.12	200
0.9	1.05	.945	190
1.0	1.0	1.0	200
1.1	0.95	1.045	200
1.2	0.9	1.08	200
1.3	0.85	1.105	200
1.4	0.7	0.98	200
1.5	0.63	0.945	200
1.6	0.57	0.912	200
1.7	0.5	0.85	200
1.8	0.45	0.81	200
1.9	0.42	0.798	200
2.0	0.4	0.8	200

The third approximation is the polynomial approximation to $e^{i2ne^{-\beta H}}$. In this application we shift the potential by a constant (the binding energy) so the spectrum of $e^{-\beta H}$ is strictly between zero and one. Then it is only necessary to find a polynomial approximation to e^{inx} for $x \in [0, 1]$.

The polynomial approximation is made using the Chebyshev expansion:

$$f(y) \approx P_N(y) := \frac{1}{2}c_0T_0(y) + \sum_{k=1}^N c_kT_k(y) \quad (8.7)$$

$$c_j = \frac{2}{N+1} \sum_{k=1}^N f(\cos(\frac{2k-1}{N+1}\frac{\pi}{2})) \cos(j\frac{2k-1}{N+1}\frac{\pi}{2}). \quad (8.8)$$

The Chebyshev expansion is designed for functions on the interval $[-1, 1]$. To make full use of the information in this approximation we approximate $e^{-2inx} = f(2x-1)$ by the polynomial approximation to $f(y)$. This gives the polynomial

$$f(2e^{-\beta H'} - 1) \approx \frac{1}{2}c_0T_0(2e^{-\beta H'} - 1) + \sum_{k=1}^N c_kT_k(2e^{-\beta H'} - 1) = P_N(e^{-\beta H'}) \quad (8.9)$$

where $H' = H + e_b$. It follows that

$$|e^{2inx} - P_N(x)| < 2\frac{n^{N+1}}{(N+1)!} \quad |||e^{2ine^{-\beta H'}} - P_N(e^{-\beta H'})||| < 2\frac{n^{N+1}}{(N+1)!} \quad (8.10)$$

Chebyshev polynomials are known to be good approximations to the best uniform polynomial approximation [22]. Table 10 shows polynomial approximations to e^{inx} for different values of $x \in [0, 1]$, and n . The errors are between 10^{-11} and 10^{-13} for the degree of the polynomial only slightly above n .

Table 10: Polynomial convergence

x	n	deg	poly error %
0.1	200	150	7.939e+00
0.1	200	200	3.276e+00
0.1	200	250	1.925e-11
0.1	200	300	4.903e-13
0.1	630	580	3.573e+00
0.1	630	630	2.069e+00
0.1	630	680	5.015e-08
0.1	630	700	7.456e-11
0.5	200	150	1.973e-13
0.5	200	200	1.627e-13
0.5	200	250	3.266e-13
0.5	630	580	1.430e-14
0.5	630	630	3.460e-13
0.5	630	680	9.330e-13
0.9	200	150	7.939e+00
0.9	200	200	3.276e+00
0.9	200	250	1.950e-11
0.9	200	300	9.828e-13
0.9	630	580	3.573e+00
0.9	630	630	2.069e+00
0.9	630	680	5.015e-08
0.9	630	700	7.230e-11

Table 10. suggests that a polynomial with degree 10-20% larger than n is needed for convergence to one part in 10^{12} ($10^{-10}\%$).

All three approximations are combined to get the approximations to the on-shell transition operator for incident momenta between 50 MeV and 2 GeV. These results are displayed in table 11. The errors are all better than .1%. The only significant source of error is the approximate factorization of the sharp momentum matrix element from the S matrix elements. This can be made as small as desired by choosing sufficiently narrow wave packets, although there is no need to improve accuracy to better than experimental resolution.

Table 11: Final calculation

$k0$	Real T	Im T	% error
0.05	2.18499e-1	-1.03160e+0	0.0982
0.1	-2.30337e-1	-4.09325e-1	0.0956
0.2	-1.01512e-1	-4.61420e-2	0.0981
0.3	-3.46973e-2	-6.97209e-3	0.0966
0.4	-1.39007e-2	-1.44974e-3	0.0997
0.5	-6.44255e-3	-3.86459e-4	0.0986
0.6	-3.34091e-3	-1.24434e-4	0.0952
0.7	-1.88847e-3	-4.63489e-5	0.0977
0.8	-1.14188e-3	-1.93605e-5	0.0965
0.9	-7.28609e-4	-8.86653e-6	0.0982
1.0	-4.85708e-4	-4.37769e-6	0.0967
1.1	-3.35731e-4	-2.30067e-6	0.0987
1.2	-2.39235e-4	-1.27439e-6	0.0968
1.3	-1.74947e-4	-7.38285e-7	0.0985
1.4	-1.30818e-4	-4.44560e-7	0.0955
1.5	-9.97346e-5	-2.76849e-7	0.0956
1.6	-7.73390e-5	-1.77573e-7	0.0992
1.7	-6.08794e-5	-1.16909e-7	0.0964
1.8	-4.85672e-5	-7.87802e-8	0.0956
1.9	-3.92110e-5	-5.42037e-8	0.0967
2.0	-3.20000e-5	-3.80004e-8	0.0966

These tests suggest that this method can be applied to compute scattering observables at the few GeV scale.

IX. SUMMARY AND CONCLUSION

In this paper we introduced a formulation of relativistic quantum mechanics that uses Euclidean Green functions or generating functionals as dynamical input. The motivation for this approach is to formulate few-body models at the few GeV scale. The models are quantum mechanical which means that they are formulated in terms of linear operators on a model Hilbert space. They can be treated using standard quantum mechanical methods. The advantages of this framework over some conventional treatments of relativistic quantum mechanics is that there is a formal relation to Euclidean Lagrangian field theory. Specifically, the quantum theory discussed in this paper becomes the quantum mechanical formulation of the field theory when the model Green functions are replaced by the Green functions of the field theory. A second advantage of this formalism is that all calculations can be performed in the Euclidean domain, without using analytic continuation, however the quantities computed are ordinary Minkowski scalar products of normalizable vectors. This permits the use of ordinary Minkowski-space methods.

The structure of the physical Hilbert space was discussed in section 1 in terms of a Euclidean generating functional for a scalar field theory. Methods to model generating functionals in terms of connected Euclidean Green functions were also discussed. Representations in terms of Euclidean Green functions were given in section 6. Examples illustrating the equivalence of the Hilbert space inner product expressed in terms of Euclidean Green functions and the standard Minkowski inner product were given in equations (6.4) and (6.5).

The Poincaré Lie algebra is realized by interpreting the real Euclidean group as a complex subgroup of the complex Lorentz group on the physical Hilbert space. Self-adjoint generators that satisfy the Poincaré commutation relations are extracted by considering infinitesimal Euclidean transformations which become infinitesimal complex Lorentz transformations on the physical Hilbert space. Expressions for the generators were given both in terms of the generating functional in section 3 and directly in terms of the Green functions in section 6.

In section four the generators introduced in sections three and six are used to construct a mass operator whose point eigenstates correspond to particles. The translation and rotation operators introduced in sections three and six are used to construct operators that create single particle states of a given sharp momentum and spin out of normalizable

mass eigenstates. These states necessarily transform irreducibly under the Poincaré group if they non-degenerate.

The operators that create single-particle states are used to construct two-Hilbert space injection operators in the Haag-Ruelle formulation of scattering. The existence of the wave operators can be checked using a generalization of the standard Cook's method [19] used in non-relativistic quantum mechanics. The two-Hilbert space wave operators are computed using time-dependent methods with the Kato-Birman invariance principle to reduce the computation to the evaluation matrix elements of polynomials in $e^{-\beta H}$. The feasibility of this method for computing scattering observables was established using an exactly solvable non-relativistic test model. This model demonstrated that it is possible to perform accurate calculations over a wide range of energies using this method. Heuristic guidelines for choosing the "time" n , temperature β , and the degree of the polynomial were established in the context of this model. Both the "time" limit and polynomial approximations were shown to be very accurate. Even though the input to the scattering theory involves Euclidean quantities, scattering emerges because the scattering states asymptotically oscillate in phase with free-particle states as the "time-parameter", n , gets large. The limits do not depend on the exponential fall-off that is used in lattice calculations. This means that calculations can be performed at relativistic energies without the complications that arise in alternative formulations of scattering involving Euclidean quantities [20]. Since the output of these calculations are wave operators, the relativistic intertwining properties provide a mechanism for performing finite Poincaré transformations on all scattering states. In addition, this also means the even perturbative approximations to the Green function or generating functional will yield unitary, Poincaré invariant approximate S -matrices.

Real model calculations are done using Euclidean Green functions. In principle the model Green functions should be Euclidean covariant, real, satisfy cluster properties and be reflection positive. Taken together these are restrictive conditions. While good models should satisfy these conditions, some models may fail to satisfy some of these conditions exactly. One may still get good approximations if small operators are responsible for the difference between the exact and approximate Green functions. These questions require additional study. A relevant observation is that because all of the calculations are done without analytic continuation, errors will be small if $|Z[f_i - \theta f_f] - Z_{exact}[f_i \theta f_j]|$ is small on a suitable domain of test functions. One does not have to worry about errors being amplified by analytic continuation because all observables are expressed directly in terms of $Z[f_i - \theta f_f]$ or the corresponding quantities in terms of Euclidean Green functions. Computational questions are also relevant. The Euclidean Green functions require the accurate evaluation of integrals with modest dimensions, $(4(N-1))$ where N is the number of particles. While the integrands are smooth, the integrals must be calculated accurately, particularly for scattering calculations. The dimensions could make a sufficiently accurate calculation of the matrix elements difficult. This also needs to be explored using better models.

Another important open question that requires more investigation is the implementation of this strategy in QCD, where reflection positivity need not hold on states that are not color singlets. For the type of calculations discussed, i.e one particle states and scattering states, the two-Hilbert space injection operator has range in the space of physical states and one expects that reflection positivity holds on this subspace, which is all that is needed to implement the computational strategies discussed in this paper. One interesting observation in QCD, which is often formulated in terms of Euclidean Green functions, is that there are no problems in dealing with relative Euclidean times between physical states in the formulation of scattering theory used in this paper. This is because the scattering is actually formulated in Minkowski space, even though the computations are performed in using Euclidean Green functions.

This work supported in part by the U.S. Department of Energy, under contract DE-FG02-86ER40286.

-
- [1] B. D. Keister and W. N. Polyzou, *Adv. Nucl. Phys.* **20**, 225 (1991).
 - [2] K. Osterwalder and R. Schrader, *Commun. Math. Phys.* **31**, 83 (1973).
 - [3] K. Osterwalder and R. Schrader, *Commun. Math. Phys.* **42**, 281 (1975).
 - [4] J. Frohlich, *Helv. Phys. Acta.* **47**, 265 (1974).
 - [5] E. P. Wigner, *Annals Math.* **40**, 149 (1939).
 - [6] J. Glimm and A. Jaffe, *Quantum Physics; A functional Integral Point of View* (Springer-Verlag, 1981).
 - [7] A. Klein and L. Landau, *J. Functional Anal.* **44**, 121 (1981).
 - [8] A. Klein and L. Landau, *Comm. Math. Phys* **87**, 469 (1983).
 - [9] J. Frohlich, K. Osterwalder, and E. Seiler, *Annals Math.* **118**, 461 (1983).
 - [10] F. Coester, *Helv. Phys. Acta* **38**, 7 (1965).
 - [11] R. Haag, *Phys. Rev.* **112**, 669 (1958).
 - [12] D. Ruelle, *Helv. Phys. Acta.* **35**, 147 (1962).
 - [13] M. Reed and B. Simon, *Methods of Modern mathematical Physics*, vol. III Scattering Theory (Academic Press, 1979).
 - [14] H. Baumgärtel and M. Wollenberg, *Mathematical Scattering Theory* (Spinger-Verlag, Berlin, 1983).
 - [15] T. Kato, *Perturbation theory for linear operators* (Spinger-Verlag, Berlin, 1966).
 - [16] C. Chandler and A. Gibson, *Indiana Journal of Mathematics.* **25**, 443 (1976).

- [17] H. Kröger, Physics. Reports. **210**, 45 (1992).
- [18] W. Brenig and R. Haag, Fort. der Physik **7**, 183 (1959).
- [19] J. M. Cook, J. Math. Phys. **36**, 1011 (1957).
- [20] L. Maiani and M. Testa, Phys. Lett. B **245**, 585 (1990).
- [21] V. Wessels and W. Polyzou, Few Body Syst. **35**, 51 (2004), nucl-th/0312004.
- [22] E. Zeidler, *Oxford User's Guide to Mathematics* (Oxford University Press, Oxford, 2004).

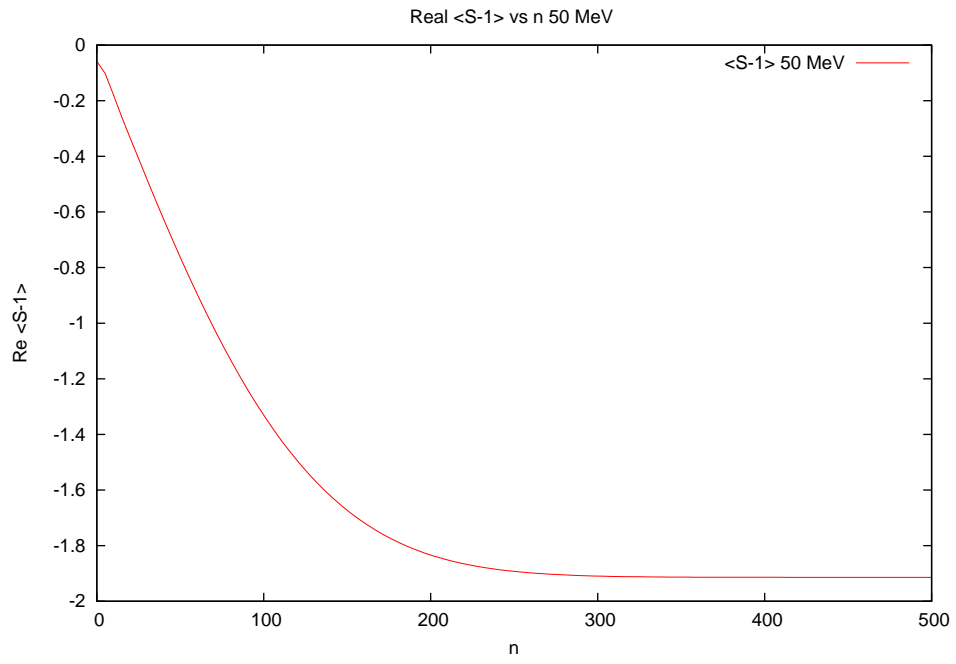


FIG. 1: $\text{Re } \langle \phi | (S - 1) | \phi \rangle$ vs n 50[MeV] $\alpha = 325000[\text{GeV}^{-2}]$

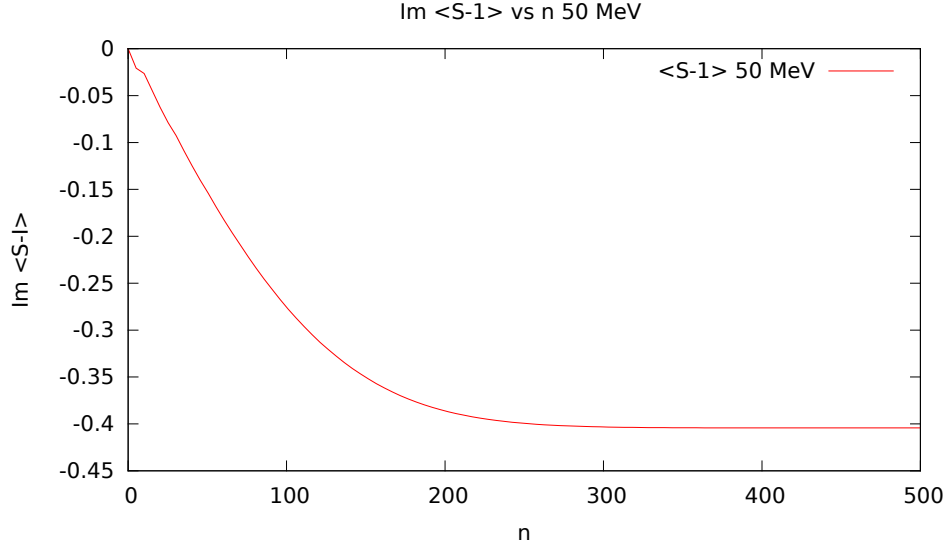


FIG. 2: $\text{Im} (\langle \phi | (S - 1) | \phi \rangle)$ vs n 50[MeV] $\alpha = 325000[\text{GeV}^{-2}]$

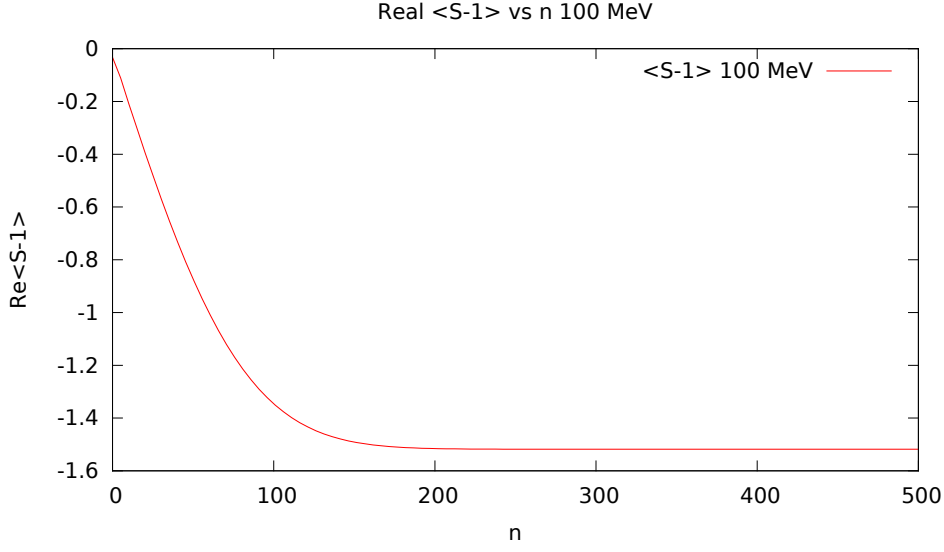


FIG. 3: $\text{Re} (\langle \phi | (S - 1) | \phi \rangle)$ vs n 100[MeV] $\alpha = 105000[\text{GeV}^{-2}]$

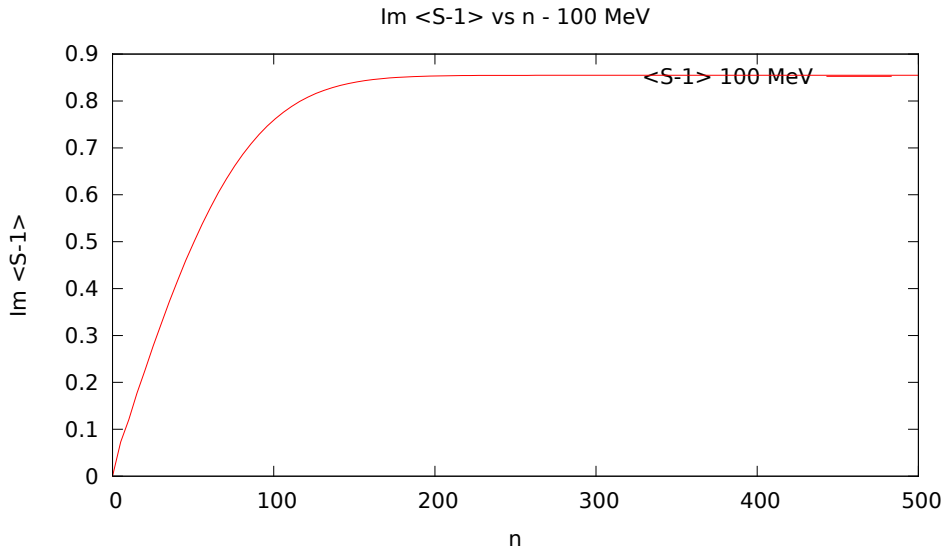


FIG. 4: $\text{Im} (\langle \phi | (S - 1) | \phi \rangle)$ vs n 100[MeV] $\alpha = 105000[\text{GeV}^{-2}]$

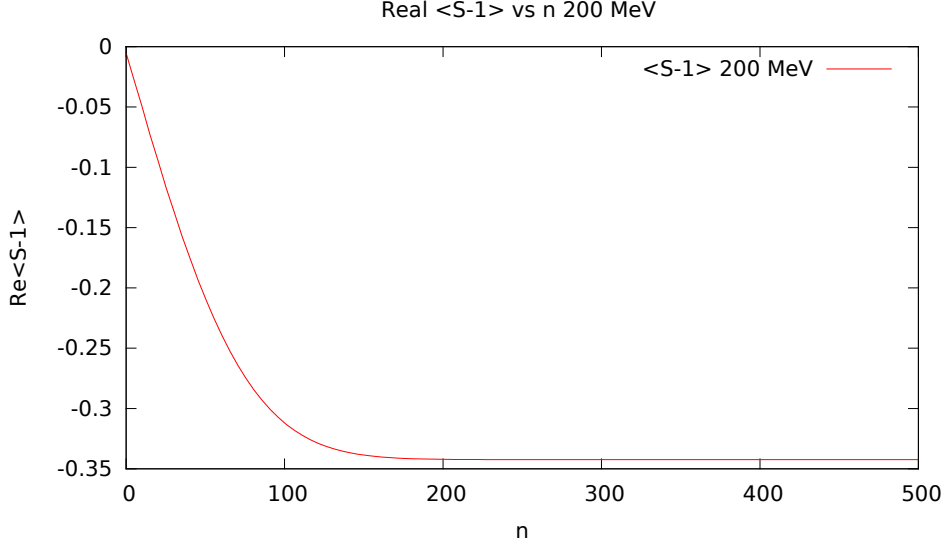


FIG. 5: $\text{Re}(\langle\phi|(S-1)|\phi\rangle)$ vs n 200[MeV] $\alpha = 26000[\text{GeV}^{-2}]$

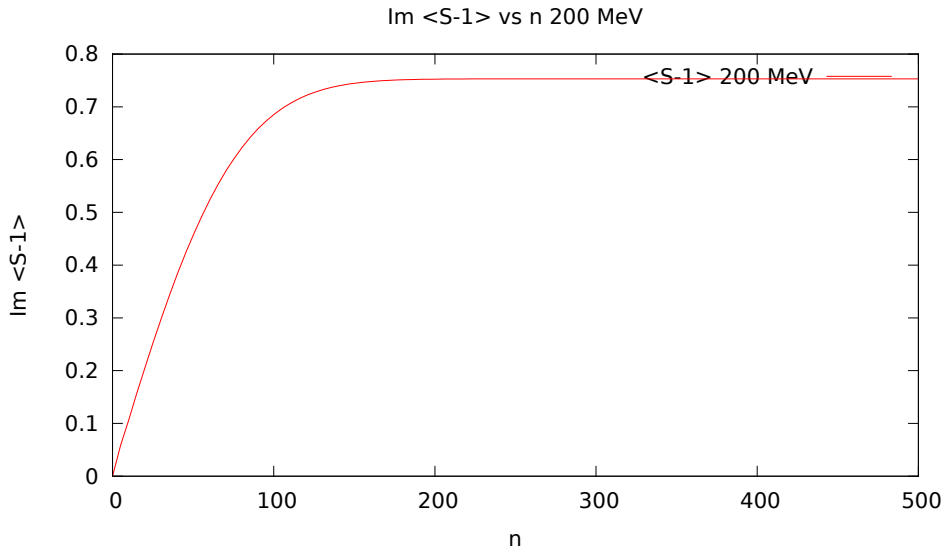


FIG. 6: $\text{Im}(\langle\phi|(S-1)|\phi\rangle)$ vs n 200[MeV] $\alpha = 32000[\text{GeV}^{-2}]$

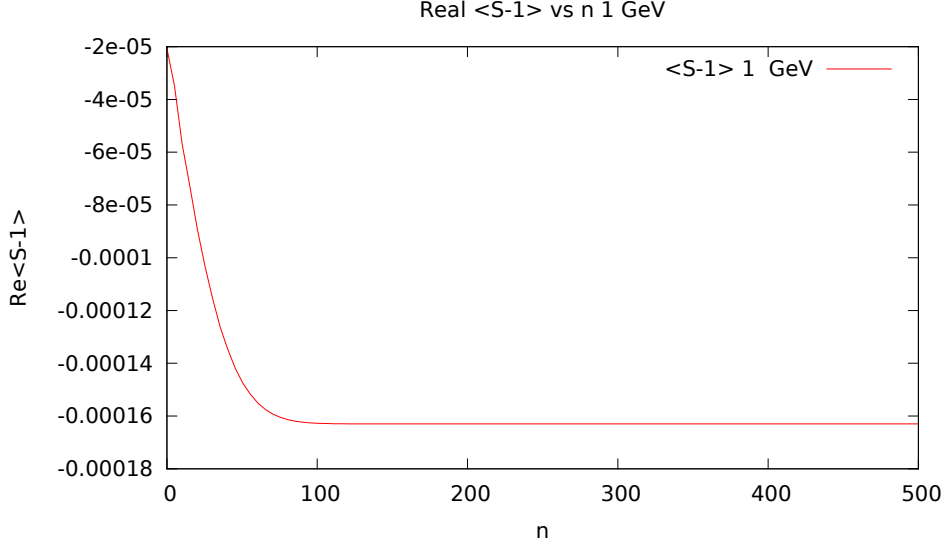


FIG. 7: $\text{Re} (\langle \phi | (S-1) | \phi \rangle)$ vs $n [1/\text{GeV}]$ $\alpha = 600 [\text{GeV}^{-2}]$

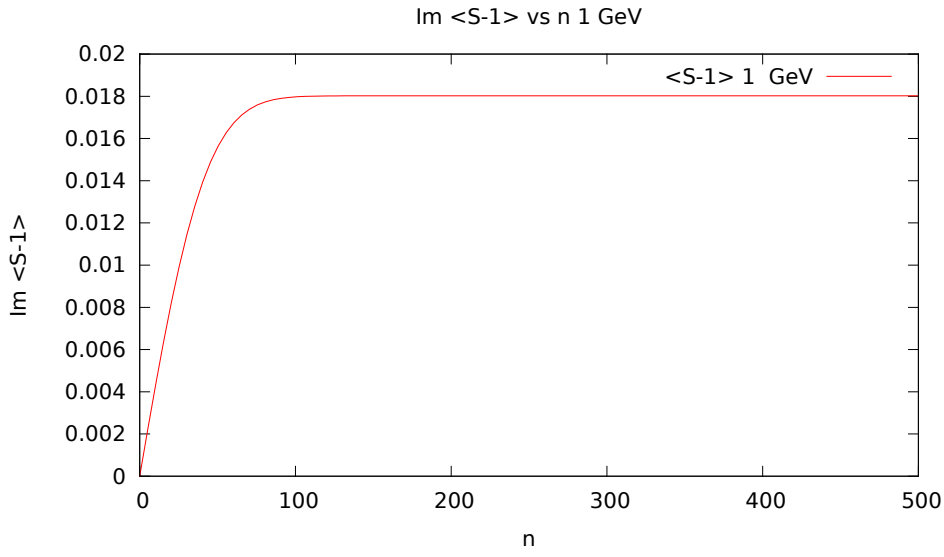


FIG. 8: $\text{Im} (\langle \phi | (S-1) | \phi \rangle)$ vs $n [1/\text{GeV}]$ $\alpha = 600 [\text{GeV}^{-2}]$

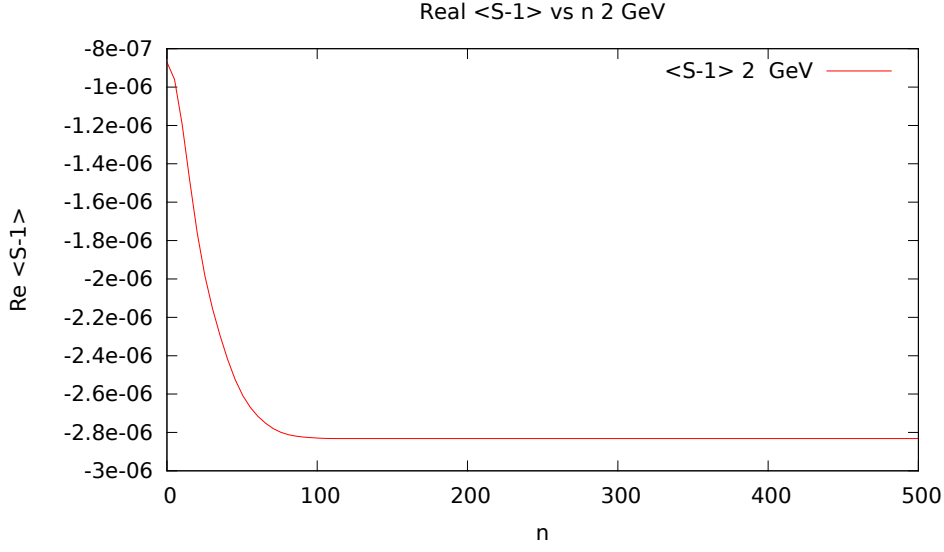


FIG. 9: $\text{Re}(\langle \phi | (S-1) | \phi \rangle)$ vs n 2[GeV] $\alpha = 250[\text{GeV}^{-2}]$

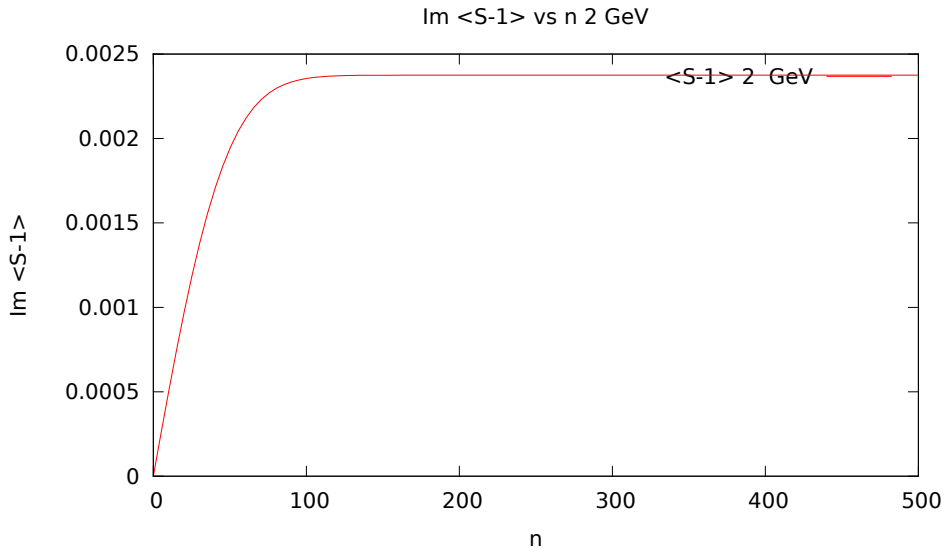


FIG. 10: $\text{Im}(\langle \phi | (S-1) | \phi \rangle)$ vs n 2[GeV] $\alpha = 250[\text{GeV}^{-2}]$

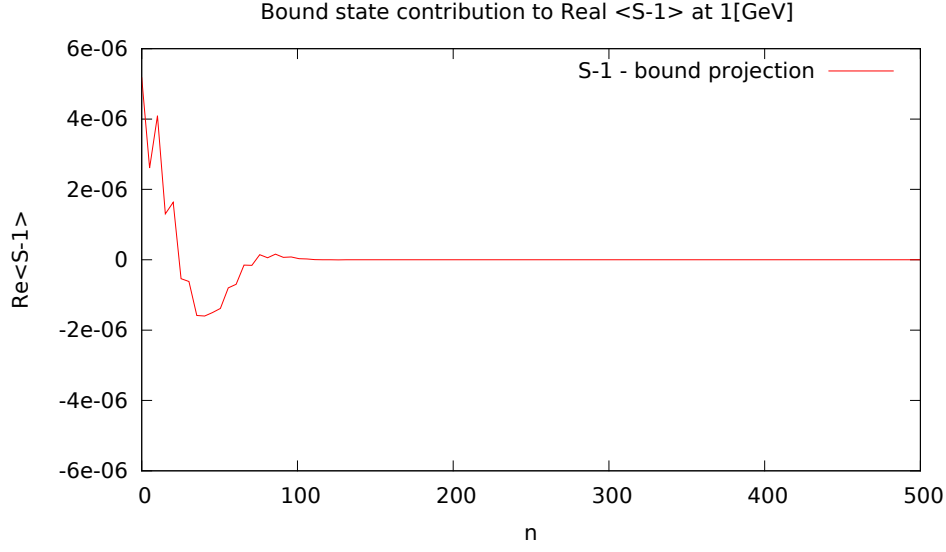


FIG. 11: Bound state contribution to $\text{Re}(\langle \phi | (S-1) | \phi \rangle)$ vs n [GeV] $\alpha = 600[\text{GeV}^{-2}]$

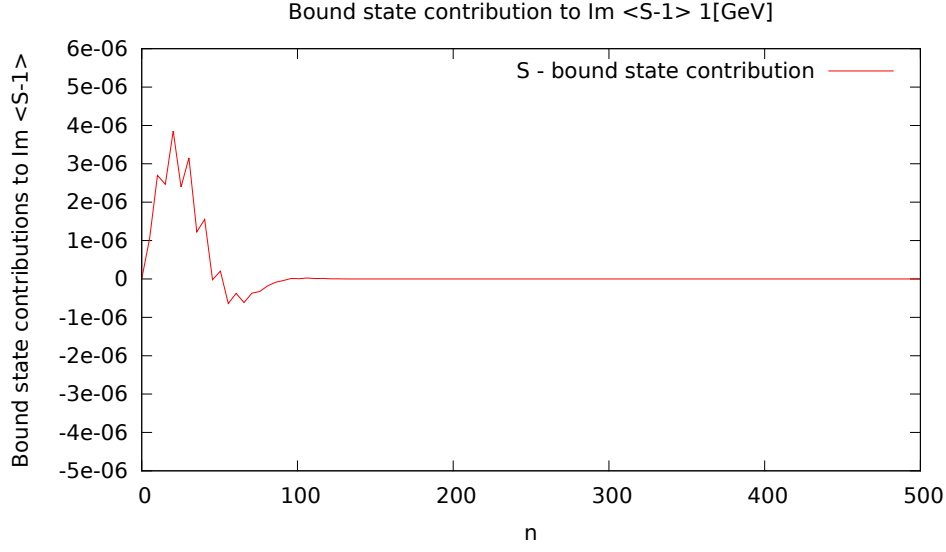


FIG. 12: Bound state contribution to $\text{Im}(\langle \phi | (S - 1) | \phi \rangle)$ vs n 1[GeV] $\alpha = 600[\text{GeV}^{-2}]$